A Primer on Unimodular Gravity

幺模引力入门

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摘要

This chapter provides an introduction to unimodular gravity both at the classical and quantum levels, discussing the rôle it might play in the partial solution of the cosmological constant problem. The main objective of this work is to serve as a conceptual introduction to unimodular gravity without disregarding computational detail. In that sense, techniques used at the research level are presented and applied in detail.

本章对经典和量子层面的幺模引力进行了介绍，探讨了它在部分解决宇宙学常数问题中可能发挥的作用。本著作的核心目标是在兼顾计算细节的同时，从概念层面介绍幺模引力。就此而言，本文呈现并详细应用了前沿研究中所用的技术。

Keywords

关键词

Unimodular gravity - Cosmological constant problem - Modified gravity

幺模引力——宇宙学常数问题——修正引力

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Introduction

引言

Unimodular gravity (UG) was first considered by Einstein in 1919 [1] as an attempt to make a connection between gravity and Mie's theory of electromagnetic wave scattering, published a few years prior. The original proposal was a far cry from the current status of UG. However, 7 years later, in his book [2], Pauli discussed the topic in a much more modern flavor. The theory took rise when the attractive properties of UG in the context of a field theory were noticed in [3], which was soon followed by the works of [4-7].

幺模引力 (UG) 最早由爱因斯坦于 1919 年提出 [1]，旨在建立引力与米氏几年前发表的电磁波散射理论之间的关联。最初的提议和幺模引力如今的研究现状相去甚远。但 7 年后，泡利在其著作 [2] 中以更贴近现代的视角讨论了该主题。当文献 [3] 发现幺模引力在场论框架下具备诸多引人关注的性质后，该理论开始兴起，随后很快便涌现出文献 [4-7] 的相关工作。

UG was originally conceived as a metric theory of gravity in which the determinant of the metric satisfies the unimodular condition:

幺模引力最初被定义为一种引力度量理论，其中度量的行列式满足幺模条件:

$$g \equiv \left|\det\left(g_{\mu\nu}\right)\right| = 1. \tag{1}$$

The general condition for a diffeomorphism (Diff in the sequel)

微分同胚 (下文简称 Diff) 的一般条件

$$x \to x', \tag{2}$$

to preserve the unimodular condition is that the Jacobian satisfies

保持幺模条件要求雅可比行列式满足

$$J \equiv \det \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} = \pm 1 \tag{3}$$

In a more modern language, UG is often introduced by considering an action functional for the metric and the matter fields, represented here by ϕ :

在更现代的表述中，幺模引力通常通过引入度量与物质场的作用泛函来介绍，此处由 ϕ 表示:

$$S[g,\phi] \equiv \int d(vol)\mathcal{L}[g,\phi] \tag{4}$$

for which the volume form is fixed, i.e.,

对于该作用泛函，体积形式是固定的，即

$$d(vol) = \omega\varepsilon_{\mu_1\ldots\mu_d} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d}, \tag{5}$$

and one then sets $\sqrt{g} = \omega$. This theory then will not be invariant under general coordinate transformations Diff but under those coordinate transformations that preserve the determinant of the metric (and hence the volume form). These are the volume-preserving These are the transverse, or volumepreserving Diffs (VPD). It is well known [8] that not all Diffs that are close to the identity can be obtained by the exponential map; nevertheless, the ones for which this is possible can be represented by a transverse vector:

随后我们设定 $\sqrt{g} = \omega$。该理论在一般坐标变换 Diff 下不具有不变性，仅在保度量行列式 (因此也保体积形式) 的坐标变换下不变。这些就是保体积变换，即横向或保体积微分同胚 (VPD)。众所周知 [8]，并非所有靠近恒等变换的微分同胚都可以通过指数映射得到；尽管如此，可以通过指数映射得到的那些微分同胚可以用横向矢量表示为:

$$\nabla_\mu \xi^\mu = 0. \tag{6}$$

It is quite easy to check that VPD generate a subgroup of all Diffs connected to the identity, Diff, because if ξ_1^μ and ξ_2^μ are transverse, then so is their commutator:

很容易验证，保体积微分同胚构成了所有连通到恒等变换的微分同胚群 Diff 的一个子群，因为如果 ξ_1^μ 和 ξ_2^μ 是横向的，那么它们的对易子也是横向的:

$$\left[\xi_1^\mu, \xi_2^\mu\right] \in VPD \tag{7}$$

In UG, the equations of motion (EM)

在幺模引力中，运动方程 (EM)

$$E_{\mu\nu} \equiv \frac{\delta S}{\delta g^{\mu\nu}}, \tag{8}$$

obey (see [12]) the following condition:

满足如下条件 (参见 [12]):

$$\nabla_{[\lambda|}\nabla^{\mu}E_{\mu|\upsilon]} = 0, \tag{9}$$

which is equivalent to the existence of a function S such that

这等价于存在函数 S 使得

$$\nabla_{\mu}E^{\mu}{}_{\nu} = \nabla_{\nu}S \tag{10}$$

as long as the de Rham cohomology of the manifold is trivial $H^3_{dR}(M) = 0$; otherwise, harmonic contributions have to be included. In fact,

只要流形的德拉姆上同调是平凡的 $H^3_{dR}(M) = 0$；否则必须纳入调和贡献。事实上,

$$\delta S = \int d\,(vol)\,E_{\mu\upsilon}\,(\nabla^{\mu}\xi^{\upsilon} + \nabla^{\upsilon}\xi^{\mu}) =$$

$$= \int d\,(vol)\,E_{\mu\nu}\left(\nabla^{\mu}\eta^{\nu\alpha\beta\gamma}\nabla_{\alpha}\Omega_{\mu\gamma} + \nabla^{\upsilon}\eta^{\mu\alpha\beta\gamma}\nabla_{\alpha}\Omega_{\beta\gamma}\right), \tag{11}$$

where

其中

$$\Omega_{\mu\nu} = -\Omega_{\nu\mu},\ \eta_{\alpha\beta\gamma\delta} \equiv \sqrt{g}\varepsilon_{\alpha\beta\gamma\delta}, \tag{12}$$

and the statement follows. Of course, in the Diff invariant case, $S =$ constant.

上述结论由此得证。当然，在微分同胚不变的情形下，$S =$ 为常数。

Alternatively to this quick presentation, section ”Linear Field Theory” closely follows the spirit of [3]. Thus, it reviews the general yet erroneous idea that general relativity (GR) is the only ghost-free theory of a massless spin-two particle.

除上述简要介绍外，“线性场论”一节严格遵循文献 [3] 的思路。该节复盘了一个普遍但错误的观点: 广义相对论 (GR) 是唯一无鬼的质量为零的自旋二粒子引力理论。

Let us precede this construction with a motivation for UG in the context of the cosmological constant problem (CCP); see [9] for the original review or [10] for an updated and extensive bibliographic guide of the topic. The aspect of the CCP considered here corresponds to the fact that contrary to Minkowski spacetime, where normal ordering can set vacuum energy to zero, on curved spacetimes, vacuum energy is naively expected to contribute to the stress-energy tensor as

在展开这一构造之前，我们先在宇宙学常数问题 (CCP) 的框架下说明幺模引力 (UG) 的研究动机；相关原始综述见文献 [9]，关于该主题更新且详尽的文献指南见文献 [10]。本文讨论的宇宙学常数问题的核心在于: 在闵氏时空中我们可以通过正规排序将真空能置零，但在弯曲时空中，按朴素的预期真空能会对能动张量产生贡献，形式为

$$\left\langle T_{\mu}^{\mu} \mid T_{\mu}^{\mu}\right\rangle_0 g_{\mu\nu}, \tag{13}$$

which has the same form as the CC term on the (EM) of GR:

该形式与广义相对论 (GR) 爱因斯坦场方程 (EM) 中的宇宙学常数项完全一致:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \lambda g_{\mu\nu} = 2\kappa^2 \left(\widetilde{T}_{\mu\nu} + \left\langle T_{\mu}{}^{\mu} \mid T_{\mu}{}^{\mu} \right\rangle_0 g_{\mu\nu} \right). \tag{14}$$

Here, $\widetilde{T}_{\mu\nu}$ represents all contributions to the stress-energy tensors except the ones coming from the vacuum expectation value of quantum fields.

在此处，$\widetilde{T}_{\mu\nu}$ 表示应力-能量张量除来自量子场真空期望值之外的所有贡献。

Therefore, in GR, the CC term and vacuum energy contributions to the stress-energy tensor have the same form.

因此，在广义相对论中，宇宙学常数项与真空能对能动张量的贡献形式完全相同。

However, any gross estimate for the vacuum energy of quantum fields yields results over 30 orders of magnitude from the observed value for the CC [9, 10] . Thus, there is a fine-tuning problem in which the different contributions to the vacuum energy, including those coming from phase transitions, have to cancel to an accuracy way beyond 30 decimal places. It is in this sense that the CC is non-natural.

然而，对量子场真空能的任何粗略估计得出的结果，与观测到的宇宙学常数值相差超过 30 个数量级 [9, 10] 。因此，这里存在一个精细调节问题: 包括相变贡献在内，不同来源对真空能的贡献必须抵消，精度要求远超过小数点后 30 位。正是在这个意义上，宇宙学常数是非自然的。

In section "Non-linear Field Theory," it is shown that, in the context of UG, the CC appears as an integration constant with no naturalness problem associated. This is followed by a discussion on the all-important problem of coupling sources to UG in section "Physical Sources." The status of Birkhoff's theorem in UG is discussed in section "No Birkhoff's Theorem in UG." Section "Unimodular Cosmology" is an invitation to unimodular cosmology. In section "Tree Diagrams," tree-level calculations for UG are presented. This is completed by the one-loop effect discussion of section "One-Loop Unimodular Gravity," where necessary tools to apply the path integral formalism to UG are introduced.

在“非线性场论”一节中我们将展示，在幺模引力的框架下，宇宙学常数是以积分常数的形式出现的，不存在相关的自然性问题。随后我们在“物理源”一节讨论了幺模引力中源耦合这个至关重要的问题。我们在“幺模引力中不存在伯克霍夫定理”一节讨论了幺模引力下伯克霍夫定理的成立情况。“幺模宇宙学”一节是对幺模宇宙学的介绍。在“树图”一节，我们给出了幺模引力的树图计算结果。随后我们在“单圈幺模引力”一节讨论了单圈效应，并在该节引入了将路径积分形式化应用于幺模引力所需的工具。

Linear Field Theory

线性场论

This section reviews the theory of a linear field that only propagates a massless spin-two field, with two polarization degrees of freedom (DOF). As originally discussed in [3] and later in [11], here it is shown that the linear Diff's invariance, LDiff, is too restrictive in the sense that the subgroup of volume-preserving linear diffeomorphisms, LTDiff, can successfully do the job.

本节回顾仅传播具有两个极化自由度 (DOF) 的无质量自旋 2 场的线性场理论。正如文献 [3] 最初讨论、文献 [11] 后续讨论的那样，本文指出线性微分同胚不变性 LDiff 的约束过强，而保体积线性微分同胚子群 LTDiff 可以恰当地满足要求。

The notion of a massless free particle in flat spacetimes is tied to the invariance under the Poincaré group (in particular to the proper orthochronous Lorentz group $\mathcal{L}_{\uparrow}^{+}$) of the unitary representation of the covering group of the little group of the four-vector

平直时空下无质量自由粒子的概念与庞加莱群 (尤其是真正时洛伦兹群 $\mathcal{L}_{\uparrow}^{+}$) 下四动量小群覆盖群的幺正表示的不变性紧密相关

$$k = (E, 0, 0, E)\,. \tag{15}$$

To make the group finite-dimensional, gauge invariance introduces the equivalence class

为使群成为有限维，规范不变性引入了等价类

$$h_{\mu\nu}(k) \equiv h_{\mu\nu}(k) + 2k_{(\mu}\xi_{\nu)}(k)\,, \tag{16}$$

with the two constraints

满足两个约束条件

$$k^2\xi_\mu(k) = 0 \;\text{ and }\; k_\mu\xi^\mu(k) = 0. \tag{17}$$

As mentioned in [3], the first condition is too restrictive for interaction, so it must be discarded. At this point, if one also drops the second condition,

正如文献 [3] 所述，第一个约束对相互作用而言限制过强，因此必须舍弃。此时，如果我们同时去掉第二个约束，

$$h_{\mu\nu}(k) \equiv h_{\mu\nu}(k) + 2k_{(\mu}\xi_{\nu)}(k)\,, \tag{18}$$

which is known to correspond to linearized diffeomorphisms LDiff. Nevertheless, from the above discussion, one can see that keeping the transverse condition does the job. This would correspond in position space to

这第二个约束对应线性化微分同胚 LDiff。但从上述讨论可以看出，保留横向条件即可满足要求。这在位置空间中对应

$$h_{\mu\nu}(x) \equiv h_{\mu\nu}(x) + 2\partial_{(\mu}\xi_{\nu)}(x)\,, \tag{19}$$

$$\partial_\mu \xi^\mu(x) = 0. \tag{20}$$

This characterizes linearized transverse diffeomorphisms, LTDiff.

这就是线性化横向微分同胚 LTDiff 的特征。

A neat physical interpretation of Eqs. (19) and (20) can be given by realizing that, from a physical perspective, LTDiff transformations have three DOF. These three DOF are enough to pass from the five DOF of massive gravity to the two DOF of a theory that only propagates a massless spin-two particle.

对式 (19) 和 (20) 可以给出清晰的物理解释: 从物理角度看，LTDiff 变换具有三个自由度。这三个自由度足以将有质量引力的五个自由度约化为仅传播无质量自旋 2 粒子理论的两个自由度。

Considering these observations, a linearized Lagrangian that is second order in derivatives and such that it is ghost-free while only propagating a spin-two massless particle can be built. Writing the more general set of operators that satisfy the above conditions,

基于这些观察，我们可以构造一个二阶导数的线性化拉格朗日量，它无鬼场且仅传播无质量自旋 2 粒子。写出满足上述条件的最广泛算符集合后，

$$\mathcal{L} \equiv \sum_{i=0}^{4} C_i O^{(i)} \tag{21}$$

where defining $h \equiv \eta^{\alpha\beta} h_{\alpha\beta} = h_\alpha{}^\alpha$. [1]

其中定义 $h \equiv \eta^{\alpha\beta} h_{\alpha\beta} = h_\alpha{}^\alpha$. [1]

$$\mathcal{O}^{(1)} \equiv \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} \;\; \mathcal{O}^{(2)} \equiv -\frac{1}{2}\partial_\lambda h^{\mu\lambda}\partial_\rho h_\mu{}^\rho \tag{22}$$

$$\mathcal{O}^{(3)} \equiv \frac{1}{2}\partial_\mu h \partial_\lambda h^{\mu\lambda} \;\; \mathcal{O}^{(4)} \equiv -\frac{1}{4}\partial^\mu h \partial_\mu h \tag{23}$$

Setting $C_1 = 1$ for normalization and requiring invariance under LTDiff yields [2]

取 $C_1 = 1$ 做归一化并要求满足 LTDiff 不变性，可得 [2]

$$C_1 = C_2 = 1\,. \tag{24}$$

Additionally, imposing invariance under Weyl transformations results in

此外，要求外尔变换不变性会得到

$$C_3 = \frac{2}{n} \text{ and } C_4 = \frac{n+2}{n^2}. \tag{25}$$

---

[1] When the background metric $\bar{g}_{\alpha\beta}$ is arbitrary, this will generalize to $h \equiv \bar{g}^{\alpha\beta} h_{\alpha\beta}$. [2] For Fierz-Pauli, $C_i = 1$. Thus, LTDiff gives a less stringent condition.

[1] 当背景度规 $\bar{g}_{\alpha\beta}$ 任意时，这将推广为 $h \equiv \bar{g}^{\alpha\beta} h_{\alpha\beta}$ 。[2] 对于菲尔兹-泡利理论，$C_i = 1$ 。因此，LTDiff 给出的约束更宽松。

LWTDiff [3] can be obtained by combining Eqs. (21),(24), and (25).

LWTDiff [3] 可通过结合式 (21)、(24) 和 (25) 得到。

It is interesting to note that the same constraints for C_i could have been obtained by starting from the Fierz-Pauli Lagrangian (with $C_i = 1$) rewriting

值得注意的是，对 C_i 的相同约束也可以从菲尔兹-泡利拉格朗日量 (带有 $C_i = 1$) 出发，通过重写得到

$$h_{\mu\upsilon} \to h_{\mu\upsilon} - \frac{1}{n} h \eta_{\mu\upsilon} \tag{26}$$

Note that Eq. (26) does not represent a field redefinition, as it is not invertible.

请注意，式 (26) 并不代表场重新定义，因为它不可逆。

In the next section, the fully non-linear theory is developed, where LTDiff is substituted by TDiff. Recall that TDiff correspond to VPD connected to the identity.

在下一节中，我们将构建完整的非线性理论，其中 LTDiff 被替换为 TDiff。回顾可知，TDiff 对应连通到恒等元的 VPD。

Non-linear Field Theory

非线性场论

In the previous section, the theory at the linear level was considered. The complete, non-linear theory can be obtained by considering the Einstein-Hilbert action for a unimodular metric $\gamma_{\mu\nu}$:

在前一节中，我们讨论了线性层面的理论。完整的非线性理论可以通过研究幺模度规 $\gamma_{\mu\nu}$ 的爱因斯坦-希尔伯特作用量得到:

$$S\left[\hat{g}_{\mu\nu}\right]=-\frac{1}{2\kappa^2}\int d^n x\sqrt{\gamma}R\left[\gamma_{\mu\nu}\right]. \tag{27}$$

The EM can be obtained considering only transverse variations of the metric [12]. Consider for the moment a practical approach that will prove useful when quantizing. For this, define an auxiliary field from the unimodular metric

爱因斯坦方程可以仅通过考虑度规的横向变分得到 [12]。我们现在讨论一种在量子化时非常有用的实用方法。为此，我们从幺模度规定义一个辅助场

$$\gamma_{\mu\nu}\rightarrow g_{\mu\nu}g^{-\frac{1}{n}} \tag{28}$$

Since Eq. (27) is invariant only under volume-preserving diffeomorphism, for the allowed changes of coordinates, the determinant of the metric will behave as a scalar and not as a density. Then, Eq. (28) can be regarded as a conformal (or Weyl) transformation:

由于式 (27) 仅在保体积微分同胚下不变，对于允许的坐标变换，度规的行列式表现为标量而非密度。因此式 (28) 可以视为共形 (即外尔) 变换:

$$\gamma_{\mu\nu}\rightarrow e^{2\phi(x)}g_{\mu\nu}. \tag{29}$$

As previously noted, Eq. (28) is not a field redefinition because it is not invertible. Actually, the variation can be written in terms of $\delta\hat{g}_{\mu\nu}$:

如前所述，式 (28) 不是场重新定义，因为它不可逆。实际上，变分可以用 $\delta\hat{g}_{\mu\nu}$ 写为:

$$\delta\gamma_{\mu\nu}\equiv M^{\alpha\beta}_{\mu\nu}\delta g_{\alpha\beta}=g^{-\frac{1}{n}}\left(\frac{1}{2}\left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}+\delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}\right)-\frac{1}{n}g^{\alpha\beta}g_{\mu\nu}\right)\delta g_{\alpha\beta}. \tag{30}$$

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[3] The reason to consider LWTDiff will be discussed in section ”Non-linear Field Theory.”

[3] 我们会在“非线性场论”一节讨论引入保体积微分同胚的原因。

It is impossible to recover the generic metric out of Einstein’s unimodular metric (the converse is, of course, trivial). This action in an arbitrary frame is invariant under local Weyl transformations of the form of Eq. (29).

我们无法从爱因斯坦的幺模度规还原出一般度规 (当然反过来是平凡的)。该作用量在任意参考系下都对式 (29) 形式的局域外尔变换保持不变。

The action (27), after integration by parts, [4] reads in terms of $g_{\mu\nu}$

经过分部积分后，作用量 (27) [4] 可以用 $g_{\mu\nu}$ 写为

$$S\left[g_{\mu\upsilon}\right] = -\frac{1}{2\kappa^2}\int d^n x g^{\frac{1}{n}}\left(R\left[g_{\mu\upsilon}\right] + \frac{(n-1)(n-2)}{4n^2}\frac{\nabla_\mu g \nabla^\mu g}{g^2}\right). \tag{31}$$

The gauge group for the action (31) is given (see [13]) by the semi-direct product

作用量 (31) 的规范群由半直积给出 (参见 [13])

$$\text{WTDiff} = \text{Weyl} \times \text{TDiff}. \tag{32}$$

The equations of motion for this action read [5] [14]

该作用量的运动方程为 [5] [14]

$$R_{\mu\upsilon} - \frac{1}{n}Rg_{\mu\upsilon} + \frac{(2-n)(2n-1)}{4n^2}\left(\frac{\nabla_\mu g \nabla_\upsilon g}{g^2} - \frac{1}{n}\frac{(\nabla g)^2}{g^2}g_{\mu\upsilon}\right) +$$

$$+\frac{n-2}{2n}\left(\frac{\nabla_\mu \nabla_\upsilon g}{g} - \frac{1}{n}\frac{\nabla^2 g}{g}g_{\mu\upsilon}\right) = 2\kappa^2\left(T_{\mu\upsilon} - \frac{1}{n}Tg_{\mu\upsilon}\right), \tag{33}$$

Choosing the gauge $g = 1$, Eq. (33) becomes the traceless EM of GR [1,15]:

选取规范 $g = 1$ 后，式 (33) 变为广义相对论的无迹爱因斯坦方程 [1,15]:

$$R_{\mu\upsilon} - \frac{1}{n}Rg_{\mu\upsilon} = 2\kappa^2\left(T_{\mu\upsilon} - \frac{1}{n}Tg_{\mu\upsilon}\right). \tag{34}$$

Now, in Eq. (34), there seems to be no CC. However, assuming the covariant conservation of the stress-energy tensor6 and using the Bianchi identities,

现在，式 (34) 中看似不存在宇宙学常数。然而，假设应力能量张量满足协变守恒 6，并利用比安基恒等式,

$$\nabla_\mu\left(T^{\mu\upsilon} - \frac{1}{n}Tg^{\mu\upsilon}\right) = -\frac{1}{n}\nabla^\upsilon T \rightarrow \tag{35}$$

$$\nabla^\mu\left(R_{\mu\upsilon} - \frac{1}{n}Rg_{\mu\upsilon}\right) = \frac{n-2}{2n}\nabla^\upsilon R = -\frac{2\kappa^2}{n}\nabla^\upsilon T. \tag{36}$$

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[4] Usually, the integral of a covariant derivative vanishes because it can be written as

[4] 通常，协变导数的积分等于零，因为它可以写为

$$\overline{\nabla}_\mu V^\mu = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}V^\mu\right),$$

so that

因此

$$\int d^n x \sqrt{g}\overline{\nabla}_\mu V^\mu = \int d^n x \partial_\mu \left(\sqrt{g} V^\mu\right) = 0,$$

assuming vanishing physical effects at the boundary. This is not true anymore with the unimodular measure. What can be written instead is

假设边界处物理效应为零。在幺模测度下这一点不再成立，我们可以将其改写为

$$\int d^n x \bar{g}^{\frac{1}{n}} \overline{\nabla}_\mu V^\mu = \frac{n-2}{n} \int d^n x V^\mu \bar{g}^{\frac{2-n}{n}} \partial_\mu \bar{g}$$

[5] Here, a stress-energy tensor has been introduced for completeness. Note also that $T \equiv T_{\mu\nu} g^{\mu\nu}$. Sources are properly dealt with in section "Physical Sources."

[5] 为了完整性我们在此引入了应力能量张量。还需要注意 $T \equiv T_{\mu\nu} g^{\mu\nu}$ 。源的相关处理会在“物理源”一节详细讨论。

[6] The validity of Eq. (35) is proven in section "Physical Sources."

[6] 式 (35) 的有效性已在“物理源”一节中证明。

In four dimensions,

在四维情形下，

$$\nabla^\mu \left(2\kappa^2 T + R\right) = 0 \rightarrow 2\kappa^2 T + R = -C. \tag{37}$$

Plugging in the constraint Eq. (37) into the gauge fixed, four-dimensional EM,

将约束式 (37) 代入规范固定后的四维爱因斯坦-麦克斯韦理论，

$$R_{\mu\upsilon} - \frac{1}{2} R g_{\mu\upsilon} + C g_{\mu\upsilon} = 2\kappa^2 T_{\mu\upsilon}. \tag{38}$$

In Eq. (38), the reader can appreciate the different context in which a cosmological constant term appears in the EM as in contrast to GR. Here, it is an integration constant term, agnostic to the vacuum expectation value of fields. This discussion will be continued when considering its stability under radiative corrections in section "One-Loop Unimodular Gravity."

读者可从式 (38) 中看出，宇宙学常数项出现在爱因斯坦-麦克斯韦理论中的情形与广义相对论不同。此处它是一个积分常数项，与场的真空期望值无关。我们将在“单圈幺模引力”一节讨论它在辐射修正下的稳定性时继续展开这一讨论。

Physical Sources

物理源

The previous discussion only considered the metric without any matter content. The subtle issue of coupling matter fields to UG is now considered in what follows. Only scalar matter is considered for simplicity. The action of a scalar field minimally coupled to UG reads

之前的讨论仅考虑了不包含任何物质的度规。下文将讨论将物质场耦合到幺模引力 (UG) 这一棘手问题。为简便起见，我们仅考虑标量物质。最小耦合到幺模引力的标量场的作用量为

$$
\begin{aligned}
S \equiv S_g + S_m = -\frac{1}{2\kappa^2}\int d^n x g^{\frac{1}{n}}\left(R\left[g_{\mu\nu}\right] + \frac{(n-1)(n-2)}{4n^2}\frac{\nabla_\mu g \nabla^\mu g}{g^2}\right) + \\
+ \int d^n x \left[g^{\frac{1}{n}}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right) - V(\phi)\right].
\end{aligned}
\tag{39}
$$

Defining $2\kappa^2 = M^{n-2}$, Eq. (33) can be expressed as

定义 $2\kappa^2 = M^{n-2}$ ，式 (33) 可写为

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{n}Rg_{\mu\nu} \equiv M^{2-n}\left(J^g_{\mu\nu} + J^m_{\mu\nu}\right) = \\
\frac{(n-2)(2n-1)}{4n^2}\left(\frac{\nabla_\mu g \nabla_\upsilon g}{g^2} - \frac{1}{n}\frac{(\nabla g)^2}{g^2}g_{\mu\upsilon}\right) \\
-\frac{n-2}{2n}\left(\frac{\nabla_\mu \nabla_\upsilon g}{g} - \frac{1}{n}\frac{\nabla^2 g}{g}g_{\mu\upsilon}\right) + \\
+\frac{M^{2-n}}{2}\left(\partial_\mu\phi\partial_\upsilon\phi - \frac{1}{n}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi g_{\mu\upsilon}\right).
\end{aligned}
\tag{40}
$$

Besides the matter part, $J^m_{\mu\nu}$, a gravitational piece $J^g_{\mu\nu}$ coming from the spacetime dependence of the metric determinant has been included as a source. This is such that sources in the matter fields are defined according to

除物质部分 $J^m_{\mu\nu}$ 外，还包含了一个来自度规行列式时空依赖的引力项 $J^g_{\mu\nu}$ 作为源。据此，物质场中的源可按如下方式定义

$$
g^{\frac{1}{n}} J_m^{\mu\upsilon} \equiv \frac{\delta S_m}{\delta g_{\mu\upsilon}}. \tag{41}
$$

Similarly, for the gravitational piece,

类似地，对于引力部分，

$$
g^{\frac{1}{n}}\left(R_{\mu\upsilon} - \frac{1}{n}Rg_{\mu\upsilon} + M^{2-n}J^g_{\mu\upsilon}\right) \equiv \frac{\delta S_g}{\delta g_{\mu\upsilon}}. \tag{42}
$$

In this work, the standard definition of the energy-momentum tensor in GR is used. For a scalar field with minimal coupling, this means

本文采用广义相对论 (GR) 中能量动量张量的标准定义。对于最小耦合的标量场，这意味着

$$T_{\mu\upsilon} \equiv \partial_\mu\phi\partial_\upsilon\phi - \left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right)g_{\mu\upsilon}. \tag{43}$$

Its trace is

其迹为

$$T \equiv g^{\alpha\beta}T_{\alpha\beta} = nV - \frac{n-2}{2}(\nabla\phi)^2, \tag{44}$$

so that the piece of the source that depends on the scalar field is precisely

因此源中依赖标量场的部分恰好是

$$J^m_{\mu\upsilon} = T_{\mu\upsilon} - \frac{1}{n}Tg_{\mu\upsilon} = \partial_\mu\phi\partial_\upsilon\phi - \frac{1}{n}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi g_{\mu\upsilon}, \tag{45}$$

and does not include the potential energy.

且不包含势能。

Let us revisit the covariant conservation equation used to prove (38):

我们再回顾一下用于证明式 (38) 的协变守恒方程:

$$\nabla_\mu\left(T^{\mu\upsilon} - \frac{1}{n}Tg^{\mu\upsilon}\right) = -\frac{1}{n}\nabla^\upsilon T. \tag{46}$$

This can be shown to hold in general precisely owing to the Ward identities of the area-preserving diffeomorphism plus Weyl symmetry. To be specific, the TDiff Ward identities guarantee that there is a function Θ such that

可以证明，该式普遍成立，这正是得益于保面积微分同胚的沃德恒等式外加外尔对称性。具体来说，切微分同胚 (TDiff) 沃德恒等式保证存在函数 Θ 满足

$$\nabla_\mu\left(g^{\frac{1}{n}-\frac{1}{2}}\frac{\delta S}{\delta g_{\mu\upsilon}}\right) = \nabla^\upsilon\Theta \tag{47}$$

The variation of the action under a variation of the scalar field reads [7]

作用量对标量场变分的结果为 [7]

$$\delta S_m = \int d^n x g^{\frac{1}{n}} g^{\mu\upsilon}\partial_\upsilon\delta\phi - V'\delta\phi = -\int d^n x\left(\partial_\upsilon\left(g^{\frac{1}{n}}g^{\mu\upsilon}\partial_\mu\phi\right) + V'\right)\delta\phi =$$

$$= \int d^n x g^{\frac{1}{n}} \left(\nabla^2 \phi - \frac{n-2}{2n} \nabla \phi \cdot \frac{\nabla g}{g} \right) + V'(\phi). \tag{48}$$

---

[7] Here, the prime denotes differentiation with respect to the scalar field.

[7] 此处，撇号表示对标量场求导。

Note that the EM of the scalar field have changed their usual form, and now they read

注意，该标量场的能量动量张量已偏离了通常的形式，现在其形式为

$$\nabla^2 \phi + g^{-\frac{1}{n}} V'(\phi) = \frac{n-2}{2n} \frac{\nabla \phi \cdot \nabla g}{g}. \tag{49}$$

It is then not difficult to check that

不难验证

$$\nabla_\mu \left(g^{\frac{2-n}{2n}} \left(\nabla^\mu \phi \nabla^\upsilon \phi - \frac{1}{n} (\nabla \phi)^2 g^{\mu \upsilon} \right) \right) = g^{\frac{2-n}{2n}} \nabla^\upsilon \left(V + \frac{n-2}{2n} (\nabla \phi)^2 \right) =$$

$$= g^{\frac{2-n}{2n}} \frac{1}{n} \nabla^\upsilon T \tag{50}$$

which modulo EM implies the desired result. Indeed, for any arbitrary constant value of λ ,

在除去能量动量张量后就能得到预期结果。事实上，对于 λ 的任意常数值,

$$\nabla_\mu \left(g^\lambda \frac{\delta S}{\delta g_{\mu \upsilon}} \right) = \left(\nabla_\mu g^\lambda \right) \frac{\delta S}{\delta g_{\mu \upsilon}} + g^\lambda \nabla_\mu \frac{\delta S}{\delta g_{\mu \upsilon}}. \tag{51}$$

Note that the additional term is multiplied by the EM.

请注意，额外项会乘以 EM。

The remainder of this section will be devoted to showing that, although the only allowed source of the gravitational field in UG is just the traceless piece of the energy-momentum tensor

本节剩余部分将证明: 尽管幺模引力 (UG) 中引力场唯一允许的源只是能量动量张量的无迹部分

$$J_{\mu \upsilon} \equiv T_{\mu \upsilon} - \frac{1}{n} T g_{\mu \upsilon}, \tag{52}$$

the full free energy produced by arbitrary sources (not only static ones) is identical to the one predicted by GR. This encompasses all weak field tests of gravitation. It is then convenient to restrict attention to the quadratic action corresponding to fluctuations around a given flat background coupled to arbitrary external sources; see [16].

任意源 (不只是静态源) 产生的总自由能与广义相对论 (GR) 预测的完全一致。这涵盖了所有引力弱场检验。因此，我们可以方便地仅讨论对应于给定平直背景涨落、耦合到任意外源的二次作用量；参见文献 [16]。

To deal with an arbitrary background, a detailed analysis of the equations of motion will be performed to compare them to those of GR. The main conclusion here is that the UG ones are a subset of the ones corresponding to GR, with the source restricted to its traceless piece and with a vanishing CC. This refines a classical analysis [12]. There, it was argued that the Bianchi identities of UG imply a first integral of the equations of motion, which, once used, made the unimodular theory fully equivalent to GR with an arbitrary CC. Here a stronger result is proven, namely, that this CC must vanish at the level of external sources.

为了处理任意背景，本文将对运动方程做详细分析，并与广义相对论的运动方程对比。此处的主要结论是: 幺模引力的运动方程是广义相对论运动方程的子集，其源限制为无迹部分，且宇宙学常数 (CC) 为零。这完善了经典分析 [12]。文献 [12] 中曾提出，幺模引力的比安基恒等式隐含运动方程的一个第一积分，代入后可证明幺模引力理论与带任意宇宙学常数的广义相对论完全等价。本文证明了一个更强的结论: 在外源层面，该宇宙学常数必须为零。

Since they will play a crucial rôle in what follows, a slight detour is now taken to derive the Bianchi identities from the field theory viewpoint. Start by defining a one form $\xi_1 \equiv \xi_\mu dx^\mu$ as well as two forms $\Omega_2 \equiv \frac{1}{2}\Omega_{\mu\upsilon} dx^\mu \wedge dx^\upsilon$. The transversality condition now reads

由于它们在后续推导中发挥着关键作用，我们现在稍作偏移，从场论的角度推导比安基恒等式。首先定义一个一元形式 $\xi_1 \equiv \xi_\mu dx^\mu$ 以及两个二元形式 $\Omega_2 \equiv \frac{1}{2}\Omega_{\mu\upsilon} dx^\mu \wedge dx^\upsilon$ 。横截性条件现在可写为

$$\xi_1 = -2\delta\Omega_2, \tag{53}$$

where the codifferential is the adjoint operator of the exterior derivative. Acting on two forms,

其中余微分是外导数的伴随算子，作用在 2 形式上,

$$\delta \equiv *^{-1} d * \tag{54}$$

so that its components obey

因此它的分量满足

$$(\delta\Omega_2)_\rho = -\frac{1}{2}\nabla^\upsilon (\Omega_2)_{\upsilon\rho}. \tag{55}$$

The number of independent components of two forms is $\begin{pmatrix} n \\ 2 \end{pmatrix}$, but the codifferential is nilpotent $\delta^2 = 0$, so one has to withdraw the three forms

二形式的独立分量数目为 $\begin{pmatrix} n \\ 2 \end{pmatrix}$，但余微分是幂零的 $\delta^2 = 0$ ，因此必须消去三形式

$$\Omega_2 = \delta\Omega_3 \tag{56}$$

and from them, one withdraws the four forms, etc. The final counting of independent gauge parameters is

再从中减去 4 形式的贡献，依此类推。最终独立规范参数的计数为

$$\binom{n}{2} - \left(\binom{n}{3} - \left(\binom{n}{4} - \cdots\right)\right) = n - 1 \tag{57}$$

where the relationship $\sum_j (-1)^j \binom{n}{j} = 0$ has been used.

其中已经用到了关系式 $\sum_j (-1)^j \binom{n}{j} = 0$ 。

Taking into account that for antisymmetric tensors, $\Omega^{\alpha\beta} = -\Omega^{\beta\alpha} \Rightarrow \ \nabla_\alpha \nabla_\beta \Omega^{\alpha\beta} \equiv 0$

考虑到对于反对称张量，$\Omega^{\alpha\beta} = -\Omega^{\beta\alpha} \Rightarrow \ \nabla_\alpha \nabla_\beta \Omega^{\alpha\beta} \equiv 0$

$$0 = \int d^n x \left(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha\right) \frac{\delta S}{\delta g_{\alpha\beta}} = \int d^n x \left(\nabla_\alpha \nabla^\rho \Omega_{\beta\rho} + \nabla_\beta \nabla^\rho \Omega_{\alpha\rho}\right) \frac{\delta S}{\delta g_{\alpha\beta}} =$$

$$= -2 \int d^n x \sqrt{g}\, \Omega^{\rho\beta} \nabla^\rho \nabla_\alpha \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\alpha\beta}}. \tag{58}$$

Assuming that $\Omega^{\rho\beta}$ is arbitrary (they either are arbitrary or else vanishing), it follows that

假设 $\Omega^{\rho\beta}$ 是任意的 (它们要么是任意的，要么等于零)，由此可得

$$\nabla^\rho \nabla_\alpha \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\alpha\beta}} = \nabla^\beta \nabla_\alpha \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\alpha\rho}}. \tag{59}$$

This is for the vector

这对向量成立

$$\Theta^\beta \equiv \nabla_\alpha \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\alpha\beta}}, \tag{60}$$

and one has the condition

我们得到条件

$$\nabla^\rho \Theta^\beta = \nabla^\beta \Theta^\rho, \tag{61}$$

which can be integrated as

该条件可以积分得到

$$\Theta_\rho = \nabla_\rho \Phi + \gamma_\rho \tag{62}$$

where $\gamma \equiv \gamma_\rho dx^\rho$ is a harmonic form. The number of independent harmonic forms depends on the manifold's topology and is referred to as the first Betti number, $b_1(M)$, the dimension of the first cohomology group, $H^1(M)$. In case there are no harmonic forms in the spacetime manifold (which happens, in particular, if it is diffeomorphic to $\mathbb{R}^n$), this shows that the Bianchi identity is modified such that

其中 $\gamma \equiv \gamma_\rho dx^\rho$ 是一个调和形式。独立调和形式的数量取决于流形的拓扑结构，被称为第一贝蒂数 $b_1(M)$ ，即第一上同调群的维数 $H^1(M)$ 。若时空流形中不存在调和形式 (特别是当它微分同胚于 $\mathbb{R}^n$ 时就会出现这种情况)，这表明比安基恒等式会被修改为

$$\nabla^\rho \Theta^\beta = \nabla^\beta \Theta^\rho = \nabla^\beta \nabla^\rho \Phi. \tag{63}$$

i.e., still holds when integrated over the whole of spacetime with the Diff invariant measure. The Weyl invariance of the action means that

即，当用微分同胚不变测度对整个时空积分时，该式仍然成立。作用量的外尔不变性意味着

$$0 = \int d^n x w(x) g_{\alpha\beta} \frac{\delta S}{\delta g_{\alpha\beta}}, \tag{64}$$

which conveys the fact that, barring topological subtleties, the trace of the EM must be a total derivative

这表明，抛开拓扑细节不谈，能量动量张量的迹必为全导数

$$g_{\alpha\beta} \frac{\delta S}{\delta g_{\alpha\beta}} = \partial_\rho \sum^\rho. \tag{65}$$

To get the Ward identities out of the action's symmetries, the parameters also need to be independent. The standard method starts with a change of variables in the path integral expressing the expectation value of a certain monomial of fields, $X\left[g_{\mu\nu}, \psi_i\right]$, where ψ_i is a generic representation of matter fields.

要从作用量的对称性得到沃德恒等式，参数也需要是独立的。标准方法从路径积分中的变量替换开始，该路径积分表示某场单项式 $X\left[g_{\mu\nu}, \psi_i\right]$ 的期望值，其中 ψ_i 是物质场的一般表示。

$$Z \langle 0_+ | X[g, \psi_i] | 0_- \rangle \equiv e^{iW} \equiv \int \mathcal{D} g_{\mu\nu} \mathcal{D} \psi X\left[g_{\mu\nu}, \psi_i\right] e^{iS_{\text{grav}}[g] + iS_{\text{matt}}[g, \psi_i]}.$$

(66)

Namely, $g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$, this leads easily to the Ward identity

也就是说，对于 $g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ ，这可以轻易推导出沃德恒等式

$$i\left\langle 0_+ \left| \frac{\delta X[g,\psi_i]}{\delta \Omega^{\mu\upsilon}(x)} \right| 0_- \right\rangle = \nabla_\mu \nabla^\alpha \left\langle 0_+ \left| X[g,\psi_i] \frac{\delta \tilde{S}}{\delta g^{\alpha\upsilon}(x)} \right| 0_- \right\rangle - (\mu \leftrightarrow \upsilon).$$

(67)

In the particular case $X = 1$, it states that the expectation value of the classical identity should vanish. There may be, in general, quantum corrections to the naive identities, either in the form of anomalies [17] or even limit cycles [18].

在 $X = 1$ 的特殊情形下，该恒等式表明经典恒等式的期望值应当为零。一般而言，对朴素恒等式可能存在量子修正，形式可以是反常 [17]，甚至是极限环 [18]。

Free Energy with External Sources in a Flat Background

平直背景下带外源的自由能

As mentioned above, to show the equivalence of UG and GR under all the weak field tests of gravitation, this section will make use of the free energy. The study of the free energy of UG neglecting self-interaction in the presence of external sources will show that it is fully equivalent to the GR one.

如上所述，为证明 UG 与 GR 在所有引力弱场检验下等价，本节将利用自由能展开研究。研究存在外源、忽略自相互作用的 UG 自由能可以发现，它与 GR 的自由能完全等价。

First-order perturbations around a flat background correspond to

平直背景附近的一阶微扰对应

$$g_{\mu\upsilon} \equiv \eta_{\mu\upsilon} + \kappa h_{\mu\upsilon}. \tag{68}$$

Using the conventions of [11], in momentum space, the kinetic part for UG reads8

采用文献 [11] 的约定，在动量空间中，UG 的动能项为 8

$$K^U_{\mu\upsilon\rho\sigma} = \frac{1}{8}k^2\left(\eta_{\mu\rho}\eta_{\upsilon\sigma} + \eta_{\mu\sigma}\eta_{\upsilon\rho}\right) - \frac{1}{8}\left(k_\upsilon k_\sigma \eta_{\mu\rho} + \eta_{\mu\sigma}k_\upsilon k_\rho + k_\upsilon k_\rho \eta_{\mu\sigma} + k_\upsilon k_\sigma \eta_{\mu\rho}\right)$$

$$+\frac{1}{2n}\left(\eta_{\mu\upsilon}k_\rho k_\sigma + \eta_{\rho\sigma}k_\mu k_\upsilon\right) - \frac{n+2}{4n^2}k^2\eta_{\mu\upsilon}\eta_{\rho\sigma}. \tag{69}$$

This can be expressed in terms of the Barnes-Rivers projectors; see Box 1.

这可以用巴恩斯-里弗斯投影算子表示；参见框 1。

Box 1

框 1

Starting with the longitudinal and transverse projectors,

从纵向和横向投影算子出发，

$$\theta_{\alpha\beta} \equiv \eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2},\ \omega_{\alpha\beta} \equiv \frac{k_\alpha k_\beta}{k^2}. \tag{B.1}$$

They obey

它们满足

(continued)

(续)

---

[8] Compare this with the GR template, which, to this order, corresponds to the Fierz-Pauli (FP) spin-two theory. Its kinetic energy piece reads

[8] 将此与 GR 的模板对比，在该阶下，GR 对应菲尔茨-保利 (FP) 自旋 2 理论，其动能项为

$$8K_{FP}^{\mu\upsilon\rho\sigma} = k^2\left(\eta^{\mu\rho}\eta^{\upsilon\sigma} + \eta^{\mu\sigma}\eta^{\upsilon\rho} - 2\eta^{\mu\upsilon}\eta^{\rho\sigma}\right)$$

$$-\left(k^\mu k^\rho\eta^{\upsilon\sigma} + k^\upsilon k^\sigma\eta^{\mu\rho} + k^\mu k^\sigma\eta^{\upsilon\rho} + k^\upsilon k^\rho\eta^{\mu\sigma} - 2k^\mu k^\upsilon\eta^{\rho\sigma} - 2k^\rho k^\sigma\eta^{\mu\upsilon}\right).$$

Box 1 (continued)

框 1(续)

$$\theta + \omega \equiv \theta_\mu^\upsilon + \omega_\mu^\upsilon = \delta_\mu^\upsilon \equiv 1$$

$$\theta^2 \equiv \theta_\alpha^\beta\theta_\beta^\gamma = \theta_\alpha^\gamma \equiv \theta$$

$$\omega^2 \equiv \omega_\alpha^\beta\omega_\beta^\gamma = \omega_\alpha^\gamma \equiv \omega$$

$$\operatorname{tr}\theta = n - 1,$$

$$\operatorname{tr}\omega = 1\,. \tag{B.2}$$

The four-index projectors are

四指标投影算子为

$$P_2 \equiv \frac{1}{2}\left(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}\right) - \frac{1}{n-1}\theta_{\mu\nu}\theta_{\rho\sigma}$$

$$P_1 \equiv \frac{1}{2}\left(\theta_{\mu\rho}\omega_{\upsilon\sigma} + \theta_{\mu\sigma}\omega_{\upsilon\rho} + \theta_{\upsilon\rho}\omega_{\mu\sigma} + \theta_{\upsilon\sigma}\omega_{\mu\rho}\right),$$

$$P_0^s \equiv \frac{1}{n-1}\theta_{\mu\upsilon}\theta_{\rho\sigma}$$

$$P_0^w \equiv \omega_{\mu\upsilon}\omega_{\rho\sigma}$$

$$P_0^{sw} \equiv \frac{1}{\sqrt{n-1}}\theta_{\mu\upsilon}\omega_{\rho\sigma}$$

$$P_0^{ws} \equiv \frac{1}{\sqrt{n-1}}\omega_{\mu\upsilon}\theta_{\rho\sigma} \tag{B.3}$$

They obey

它们满足

$$P_i^a P_j^b = \delta_{ij}\delta^{ab}P_i^b$$

$$P_i^a P_j^{bc} = \delta_{ij}\delta^{ab}P_j^{ac},$$

$$P_i^{ab} P_j^c = \delta_{ij}\delta^{bc}P_j^{ac},$$

$$P_i^{ab} P_j^{cd} = \delta_{ij}\delta^{bc}\delta^{ad}P_j^a. \tag{B.4}$$

as well as

以及

$$\operatorname{tr} P_2 \equiv \eta^{\mu\upsilon}(P_2)_{\mu\upsilon\rho\sigma} = 0$$

$$\operatorname{tr} P_0^s = \theta_{\rho\sigma}$$

$$\operatorname{tr} P_0^w = \omega_{\rho\sigma},$$

Box 1 (continued)

框 1(续)

$$\operatorname{tr} P_1 = 0$$

$$\mathrm{tr}\, P_0^{sw} = \sqrt{n-1}\omega_{\rho\sigma}$$

$$\mathrm{tr}\, P_0^{ws} = \frac{1}{\sqrt{n-1}}\theta_{\rho\sigma},$$

$$P_2 + P_1 + P_0^w + P_0^s = \frac{1}{2}\left(\delta_\mu^\nu \delta_\rho^\sigma + \delta_\mu^\sigma \delta_\rho^\nu\right). \tag{B.5}$$

Any symmetric operator can be written as

任意对称算子都可以写为

$$K = a_2 P_2 + a_1 P_1 + a_w P_0^w + a_s P_0^s + a_\times P_0^\times, \tag{B.6}$$

(where $P_0^\times \equiv P_0^{ws} + P_0^{sw}$). Then

(其中 $P_0^\times \equiv P_0^{ws} + P_0^{sw}$)。于是

$$K^{-1} = \frac{1}{a_2}P_2 + \frac{1}{a_1}P_1 + \frac{a_s}{a_s a_w - a_\times^2}P_0^w + \frac{a_w}{a_s a_w - a_\times^2}P_0^s - \frac{a_\times}{a_s a_w - a_\times^2}P_0^\times.$$

(B.7)

Define the trace-free projector

定义无迹投影算子

$$(P_{tr})_{\rho\sigma}{}^{\lambda\delta} \equiv \frac{1}{2}\left(\delta_\rho^\lambda \delta_\sigma^\delta + \delta_\rho^\delta \delta_\sigma^\lambda\right) - \frac{1}{n}\eta_{\rho\sigma}\eta^{\lambda\delta}. \tag{B.8}$$

Then, the following relations hold:

由此可得以下关系:

$$(P_2)_{\mu\nu}{}^{\rho\sigma}(P_{tr})_{\rho\sigma}{}^{\lambda\delta} = P_2$$

$$P_0^s P_{tr} = P_0^s - \frac{n-1}{n}P_0^s - \frac{\sqrt{n-1}}{n}P_0^{sw},$$

$$P_0^w P_{tr} = P_0^w - \frac{\sqrt{n-1}}{n}P_0^{ws} - \frac{1}{n}P_0^w,$$

$$P_1 P_{tr} = P_1$$

$$P_0^{sw} P_{tr} = P_0^{sw} - \frac{\sqrt{n-1}}{n}P_0^{ws} - \frac{1}{n}P_0^w,$$

$$P_0^{ws} P_{tr} = P_0^{ws} - \frac{\sqrt{n-1}}{n}P_0^{sw} - \frac{n-1}{n}P_0^s. \tag{B.9}$$

From the properties presented in Box 1,

根据框 1 给出的性质,

$$K^U = \frac{k^2}{8}\left(2P_2 + 2P_0^s + 2P_1 + 2P_0^w\right) - \frac{k^2}{8}\left(2P_1 + 4P_0^w\right)$$

$$+\frac{k^2}{2n}\left(\sqrt{n-1}P_0^\times + 2P_0^w\right) - \frac{n+2}{4n^2}k^2\left((n-1)P_0^s + \sqrt{n-1}P_0^\times + P_0^w\right)$$

$$= k^2\left\{\frac{1}{4}P_2 - \frac{n-2}{4n^2}P_0^s - \frac{n^2-3n+2}{4n^2}P_0^w + \frac{n-2}{4n^2}\sqrt{n-1}P_0^\times\right\}. \tag{70}$$

Then, it is plain that

不难看出

$$K^{WT}_{\mu\upsilon\rho\sigma}\eta^{\rho\sigma} = 0$$

$$\xi.k = 0 \Rightarrow K_{\mu\upsilon\rho\sigma}\xi^\rho k^\sigma = 0. \tag{71}$$

To gauge fix the UG action, consider adding a gauge fixing term:

为了对幺模引力 (UG) 作用量做规范固定，考虑添加如下规范固定项:

$$\mathcal{L}_{gf} = h_{\mu\upsilon}K_{gf}^{\mu\upsilon\rho\sigma}h_{\rho\sigma}, \tag{72}$$

where using Eq. (B.4),

其中利用式 (B.4) 可得

$$K_{gf}^{\mu\nu\rho\sigma} = \frac{k^6}{4\Lambda^4}P_1. \tag{73}$$

This corresponds in position space to a gauge fixing

这在位置空间中对应于一个规范固定条件

$$\mathcal{L}_{gf} = \frac{1}{2\Lambda^4}F_\alpha^2 \equiv \frac{1}{2\Lambda^4}\left(\partial_\alpha\partial^\mu\partial^\upsilon h_{\mu\upsilon} - \Box\partial^\mu h_{\alpha\mu}\right)^2, \tag{74}$$

where Λ is an arbitrary mass scale. Let us remark that this gauge choice is admissible because it can be reached uniquely through area-preserving diffeomorphism

其中 Λ 是任意质量标度。需要说明的是，该规范选择是可行的，因为它可以通过保面积微分同胚唯一实现

$$\partial_\alpha F^\alpha = 0$$

$$\delta F_\alpha = -\Box^2 \xi_\alpha \tag{75}$$

The ghost system associated with it gets complicated because the gauge parameters are not independent, [9] more will be said regarding this fact when dealing with one-loop UG in section ”One-Loop Unimodular Gravity.” However, this issue is irrelevant for the purposes at hand, which are purely tree-level.

与之关联的鬼系统会变得复杂，因为规范参数并不独立，[9] 我们会在“单圈幺模引力”一节讨论单圈幺模引力时对此展开说明。但对于本文目前仅讨论树阶的目的而言，这个问题无关紧要。

At the end of the day,

最终，

$$4K^U_{tot} = k^2 P_2 + \left((n-1)\,m^2 - \frac{n-2}{n^2}k^2\right) P_0^s + \left(m^2 - \frac{n^2-3n+2}{n^2}k^2\right) P_0^w +$$
$$\left(m^2 + \frac{n-2}{n^2}k^2\right)\sqrt{n-1}P_0^\times + \frac{k^6}{M^4}P_1. \tag{76}$$

---

[9] A complete analysis can be found in [20].

[9] 完整分析可见文献 [20]。

Therefore, the Euclidean propagator is then given, for the gauge choice above, by

因此，对于上述规范选择，欧几里得传播子可表示为

$$k^2\Delta = P_2 + \frac{M^4}{k^4}P_1 - \frac{1}{(n-2)\,m^2}\left\{\left(m^2 - \frac{n^2-3n+2}{n^2}k^2\right)P_0^s +\right.$$
$$\left.\left((n-1)\,m^2 - \frac{n-2}{n^2}k^2\right)P_0^w - \left(m^2 + \frac{n-2}{n^2}k^2\right)\sqrt{n-1}P_0^\times\right\}. \tag{77}$$

Any coupling of the gravitational fluctuation to an external source $S_{\text{int}} = \int d^n x J_{\mu\nu} h^{\mu\nu}$ has to comply with

引力涨落与外源的任意耦合 $S_{\text{int}} = \int d^n x J_{\mu\nu} h^{\mu\nu}$ 都必须满足

$$0 = \delta S_{\text{int}} = \int d^n x J_{\mu\nu}\left(\partial_\mu \partial_\rho \Omega^\rho_\mu + \partial_\nu \partial_\rho \Omega^\rho_\nu + \omega\,(x)\,h_{\mu\nu}\right), \tag{78}$$

which requires the sources to obey both $\eta_{\mu\nu}J^{\mu\nu} = 0$ and $\partial_\mu J^{\mu\nu} = \partial^\nu T$. This means that it should be related to some conserved symmetric tensor [10] by

这要求源同时满足 $\eta_{\mu\nu}J^{\mu\nu} = 0$ 和 $\partial_\mu J^{\mu\nu} = \partial^\nu T$ 。也就是说，源应当通过下式与守恒对称张量 [10] 关联

$$J_{\mu\upsilon} \equiv T_{\mu\upsilon} - \frac{1}{n} T \eta_{\mu\upsilon}. \tag{79}$$

The free energy (or effective action) after the Gaussian functional integration reads

高斯泛函积分后的自由能 (即有效作用量) 为

$$W[J] \equiv \frac{1}{2} \int d^n x d^n y J^*_{\mu\nu}(x) \Delta^{\mu\upsilon\rho\sigma}(x, y) J_{\rho\sigma}(y) =$$

$$= \frac{1}{2} (2\pi)^{2n} \int d^n k J^*_{\mu\upsilon}(k) \Delta^{\mu\upsilon\rho\sigma}(k) J_{\rho\sigma}(k), \tag{80}$$

and using the easily proven identities,

并利用易证的恒等式,

$$(P_1 J)_{\mu\nu} = 0$$

$$\mathrm{tr}\, J P_1 J = 0\,,$$

$$(P_2)_{\mu\upsilon\rho\sigma} J^{\rho\sigma} = T_{\mu\upsilon} - \frac{1}{n-1} \theta_{\mu\upsilon} T$$

$$\mathrm{tr}\,(J P_2 J) = \left|T_{\mu\upsilon}\right|^2 - \frac{1}{n-1} |T|^2, \tag{81}$$

it conveys the fact that

它表明了以下事实:

$$W[J] = \frac{1}{2} (2\pi)^{2n} \int d^n k \left\{ J^*_{\mu\nu}(k) \frac{1}{k^2} P_2^{\mu\nu\rho\sigma}(k) J_{\rho\sigma}(k) + C(k) \left|T(k)\right|^2 \right\}, \tag{82}$$

---

[10] A priori this could be different from the usual conserved energy-momentum tensor although it will be proven to be the same.

[10] A priori，它可能不同于通常的守恒能动张量，不过后续会证明二者是等价的。

---

where

其中

$$C(k) = -\frac{1}{(n-1)(n-2) k^2}. \tag{83}$$

This yields the free energy

由此可得自由能为

$$W[T]=\frac{1}{2}(2\pi)^{2n}\int d^nk\frac{1}{k^2}\left(|J_{\mu\nu}|^2-\frac{2}{n(n-2)}|T(k)|^2\right)=$$

$$=\frac{1}{2}(2\pi)^{2n}\int d^nk\frac{1}{k^2}\left(|T_{\mu\nu}|^2-\frac{1}{n-2}|T(k)|^2\right). \tag{84}$$

This free energy (when expressed in the second form) is exactly the same as the prediction of GR, which implies that the low energy physics, and so the low energy empirical tests of UG, are the same as in GR and convey the same results. Only in the non-linear regime could some differences between both theories be found.

该自由能 (写成第二种形式时) 与广义相对论的预测完全一致，这意味着低能物理规律，也就是共形引力 (UG) 的低能经验检验，都和广义相对论一致，得出的结论也相同。只有在非线性区域才能发现两种理论的差异。

Consider as a particular example the computation of the Newtonian potential. In UG, the graviton field $h_{\mu\upsilon}$ couples to the traceless part, $\widehat{T}^{\mu\upsilon}$, of the energy-momentum tensor *ȧ* la Rosenfeld or, what is the same, the traceless part of the graviton field, $\widehat{h}_{\mu\upsilon}$, couples to the energy-momentum tensor defined à la Rosenfeld:

我们以牛顿势的计算作为具体例子。在共形引力中，引力子场 $h_{\mu\upsilon}$ 耦合于无迹部分，即能动张量 *ȧ* 的 $\widehat{T}^{\mu\upsilon}$ ，遵循罗森菲尔德定义，或者等价地说，无迹引力子场 $\widehat{h}_{\mu\upsilon}$ 耦合于罗森菲尔德定义的能动张量:

$$-\frac{\kappa}{2}\int d^4x h_{\mu\upsilon}\widehat{T}^{\mu\upsilon}=-\frac{\kappa}{2}\int d^4x\widehat{h}_{\mu\upsilon}T^{\mu\upsilon}, \tag{85}$$

where

其中

$$\widehat{T}^{\mu\upsilon}=T^{\mu\upsilon}-\frac{1}{4}T\eta^{\mu\upsilon}\ \text{ and }\ \widehat{h}_{\mu\upsilon}=h_{\mu\upsilon}-\frac{1}{4}h\eta_{\mu\upsilon}. \tag{86}$$

The Newtonian potential can then be obtained [19] from the tree-level one-graviton exchange, with transfer momentum k_μ , between two very massive scalar particles by taking the static limit: $k_\mu=(0,\mathbf{k})_\mu$. Let $\mathcal{A}_{12}$ denote the amplitude for the one-graviton exchange between two scalar particles with masses M_1 and M_2 , respectively. In UG - see Eq. (85) - one has

随后牛顿势可以通过树图阶单引力子交换得到 [19]，对两个大质量标量粒子，取转移动量为 k_μ ，再取静态极限: $k_\mu=(0,\mathbf{k})_\mu$ 。设 $\mathcal{A}_{12}$ 为质量分别是 M_1 和 M_2 的两个标量粒子之间单引力子交换的振幅，在共形引力中——见式 (85)——我们得到

$$\mathcal{A}_{12}=-i\frac{\kappa^2}{4}T^1_{\mu\nu}(p_1,p_1')<\widehat{h}^{\mu\nu}(k)\,\widehat{h}^{\rho\sigma}(-k)>T^2_{\rho\sigma}(p_2,p_2'), \tag{87}$$

where $k=p_1-p_1'=p_2'-p_2$. In the previous equation, $<\widehat{h}^{\mu\upsilon}(k)\widehat{h}^{\rho\sigma}(-k)>$ denotes the free two-point function of the traceless graviton field, and $T^i_{\mu\upsilon}(p_i,p_i')$,

其中 $k=p_1-p_1'=p_2'-p_2$。上式中，$<\widehat{h}^{\mu\upsilon}(k)\widehat{h}^{\rho\sigma}(-k)>$ 表示无迹引力子场的自由两点关联函数，且 $T^i_{\mu\upsilon}(p_i,p_i')$,

$i=1,2$, denotes the lowest order contribution to the on-shell matrix elements of the energy-momentum tensor between (on-shell) states with momentum p_i and p_i' , $i=1,2$, respectively:

$i=1,2$，表示能动张量在动量分别为 p_i 和 p_i' 的在壳态之间的在壳矩阵元的最低阶贡献，$i=1,2$ 分别为:

$$T^i_{\mu\upsilon}(p_i,p_i')=p_{i\mu}p'_{i\upsilon}+p_{i\upsilon}p'_{i\mu}+\frac{1}{2}k^2\eta_{\mu\upsilon}. \tag{88}$$

Now, for very massive particles and for $k_\mu=(0,\mathbf{k})$,

现在，对于大质量粒子且满足 $k_\mu=(0,\mathbf{k})$ 时，

$$\frac{1}{2M_i}T^i_{\mu\upsilon}(p_i,p_i')=M_i\eta^{\mu 0}\eta^{\upsilon 0},\ i=1,2 \tag{89}$$

so that, in the static limit,

因此，在静态极限下，

$$\frac{1}{2M_1 2M_2}\mathcal{A}_{12}=-i\frac{\kappa^2}{4}m_1m_2<\widehat{h}^{00}(k)\widehat{h}^{00}(-k)>, \tag{90}$$

with $k_\mu=(0,\mathbf{k})_\mu$. It is the RHS of the previous equation which must be equal to the Newtonian potential in Fourier space $V_{Nw}(\mathbf{k})$, where

其中 $k_\mu=(0,\mathbf{k})_\mu$。上式的右侧必须等于傅里叶空间中的牛顿势 $V_{Nw}(\mathbf{k})$，其中

$$V_{Nw}(\mathbf{k})=-\frac{\kappa^2}{8}\frac{M_1M_2}{\mathbf{k}^2}. \tag{91}$$

Assuming that in UG the free graviton two-point function, $<h_{\mu\nu}(k)h_{\rho\sigma}(-k)>$, corresponds to

假设共形引力中自由引力子两点关联函数 $<h_{\mu\nu}(k)h_{\rho\sigma}(-k)>$ 对应

$$\begin{aligned}\langle h_{\mu\nu}(k)h_{\rho\sigma}(-k)\rangle=\frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\mu\nu}\eta_{\rho\sigma}\right)-\frac{a(k^2)}{2k^2}\eta_{\mu\nu}\eta_{\rho\sigma}+\\+\frac{b(k^2)}{(k^2)^2}\left(k_\rho k_\sigma\eta_{\mu\nu}+k_\mu k_\nu\eta_{\rho\sigma}\right)+\frac{c(k^2)}{(k^2)^3}k_\mu k_\nu k_\rho k_\sigma,\end{aligned} \tag{92}$$

where $a(k^2), b(k^2)$, and $c(k^2)$ are arbitrary real functions. The ansatz in Eq. (92) is the most general expression consistent with Lorentz covariance, boson symmetry, the symmetry of $h_{\mu\upsilon}$, and that when one replaces in the free two-point function the tensor

其中 $a\left(k^2\right), b\left(k^2\right)$，且 $c\left(k^2\right)$ 是任意实函数。式 (92) 的假设是满足洛伦兹协变性、玻色子对称性、$h_{\mu\upsilon}$ 对称性的最一般表达式，且在自由两点关联函数中替换张量时，

$$\frac{1}{2}\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma}\right). \tag{93}$$

with the following sum over polarizations:

对偏振态求和如下:

$$\sum_{\lambda=\pm 2} \varepsilon_{\mu\nu}^{(\lambda)}\varepsilon_{\rho\sigma}^{(-\lambda)}. \tag{94}$$

Only a simple pole factor $1/k^2$ multiplies this sum, as it befits the unitarity and the fact that the theory's classical action is quadratic on the derivatives. From Eq. (92), one obtains after a little algebra

该求和前仅存在一个简单极点因子 $1/k^2$，这符合幺正性要求，也符合该理论经典作用量对导数是二次型的事实。对式 (92) 稍作代数整理即可得到

$$\left\langle \widehat{h}_{\mu\nu}(k)\,\widehat{h}_{\rho\sigma}(-k)\right\rangle = \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} + \left(-\frac{1}{2} + \frac{c\left(k^2\right)}{8}\right)\eta_{\mu\nu}\eta_{\rho\sigma}\right) +$$
$$+\frac{c\left(k^2\right)}{4\left(k^2\right)^2}\left(k_\rho k_\sigma \eta_{\mu\nu} + k_\mu k_\nu \eta_{\rho\sigma}\right) + \frac{c\left(k^2\right)}{\left(k^2\right)^3} k_\mu k_\nu k_\rho k_\sigma. \tag{95}$$

Substituting the previous result in Eq. (90) - recalling that $k_\mu = (0, \mathbf{k})_\mu$,

将上述结果代入式 (90)——注意这里 $k_\mu = (0, \mathbf{k})_\mu$,

$$-i\frac{\kappa^2}{4}m_1 m_2\left\langle \widehat{h}^{00}(k)\,\widehat{h}^{00}(-k)\right\rangle = -\frac{\kappa^2}{8}M_1 M_2\left(\frac{3}{2} + \frac{c\left(-\mathbf{k}^2\right)}{8}\right)\frac{1}{\mathbf{k}^2}. \tag{96}$$

This expression will match the Newtonian potential in (91) if, and only if, $c\left(-\mathbf{k}^2\right)$ = -4, which, by Lorentz invariance, leads to

当且仅当 $c\left(-\mathbf{k}^2\right)$ = = -4 时，该表达式与式 (91) 中的牛顿势匹配，结合洛伦兹不变性可得

$$c\left(k^2\right) = -4, \tag{97}$$

regardless of the value of k_μ . In summary, a triple pole in the $k_\mu k_\upsilon k_\rho k_\sigma$ contribution to the two-point function in Eq. (92) is needed to get the Newtonian potential right. This is what happens when one works out the propagator of UG by using the BRST technique explained in [20, 21] and discussed in section "One-Loop Unimodular Gravity" below. Notice that when $n = 4$, the propagator in Eq. (167) yields the Newtonian potential since the coefficient multiplying the contribution

与 k_μ 的取值无关。综上，要得到正确的牛顿势，式 (92) 中两点函数的 $k_\mu k_\upsilon k_\rho k_\sigma$ 贡献需要存在一个三极点。这正是使用 [20, 21] 中介绍、下文” 单圈幺模引力” 一节讨论的 BRST 方法计算 UG 传播子时得到的结果。注意当 $n = 4$ 时，式 (167) 中的传播子可以给出牛顿势，因为对应贡献的乘积系数

$$\frac{k_\mu k_\nu k_\rho k_\sigma}{k^6} \tag{98}$$

is -4, at $n = 4$.

在 $n = 4$ 处为-4。

External Sources with a General Background

具有一般背景的外源

In the presence of vacuum energy, Minkowski space is not a solution. Hence, it is not an appropriate background. This section considers an arbitrary background $g_{\mu\nu}$ instead. Therefore, a closed form for the free energy will not be available in this case. The analysis here will rely instead on a detailed analysis of the field equations endowed with arbitrary sources. The conclusion will again be that the unimodular EM are equivalent to the GR ones with vanishing CC. Expanding the Lagrangian in Eq. (31) up to the second order in the metric perturbation around an arbitrary background,

存在真空能量时，闵氏空间不是解，因此不是合适的背景。本节转而考虑任意背景 $g_{\mu\nu}$ 。这种情况下无法得到自由能的闭式解，我们的分析将转而依赖对带有任意源的场方程的细致讨论。结论依旧是: 幺模爱因斯坦场方程等价于宇宙学常数为零的广义相对论爱因斯坦场方程。将拉格朗日量在式 (31) 中围绕任意背景按度规微扰展开至二阶，

$$g_{\mu\upsilon} = g_{\mu\upsilon} + \kappa h_{\mu\upsilon}, \tag{99}$$

reads

形式为

$$\mathcal{L}_U \equiv g^{\frac{1}{n}} R + \frac{(n-1)(n-2)}{4n^2} \frac{(\nabla g)^2}{g^2} =$$

$$\bar{g}^{\frac{1}{n}} \left[\bar{R} + \kappa \left(\frac{h}{n} \bar{R} - h^{\alpha\beta} \bar{R}_{\alpha\beta} - \overline{\nabla}^2 h + \overline{\nabla}_\alpha \overline{\nabla}_\beta h^{\alpha\beta} \right) \right.$$

$$+ \frac{\kappa^2}{4} \left\{ -2\overline{\nabla}_\beta \left(h^{\alpha\rho} \overline{\nabla}^\beta h_{\alpha\rho} \right) + 2\overline{\nabla}_\alpha \left(2h^{\alpha\rho} \overline{\nabla}^\beta h_{\rho\beta} - h^{\alpha\rho} \overline{\nabla}_\rho h \right) \right.$$

$$-2h^{\beta\upsilon} \left(\overline{\nabla}^\rho \overline{\nabla}_\upsilon h_{\rho\beta} + \overline{\nabla}^\rho \overline{\nabla}_\beta h_{\rho\upsilon} - \overline{\nabla}^2 h_{\upsilon\beta} - \overline{\nabla}_\beta \overline{\nabla}_\upsilon h \right) - 2\overline{\nabla}_\sigma h \overline{\nabla}^\sigma h +$$

$$+2\overline{\nabla}_\sigma h \overline{\nabla}_\beta h^{\sigma\beta} - 2\overline{\nabla}_\sigma h_{\alpha\upsilon} \overline{\nabla}^\upsilon h^{\alpha\sigma} + \overline{\nabla}_\sigma h_{\alpha\upsilon} \overline{\nabla}^\sigma h^{\alpha\upsilon} + 4h^\beta_\alpha h^{\upsilon\alpha} \bar{R}_{\upsilon\beta} -$$

$$-\frac{4}{n}h\left(h^{\alpha\beta}\bar{R}_{\alpha\beta}+\overline{\nabla}^2 h-\overline{\nabla}_\alpha\overline{\nabla}_\beta h^{\alpha\beta}\right)+\frac{2}{n}\left(\frac{h^2}{n}-h^{\alpha\beta}h_{\alpha\beta}\right)\bar{R}\Big]\Big]+$$

$$+\frac{(n-1)(n-2)}{4n^2}\Bigg\{\kappa^2\Bigg(-2h^{\mu\upsilon}\overline{\nabla}_\mu h\frac{\overline{\nabla}_\upsilon\bar{g}}{\bar{g}}+\left(\overline{\nabla}h\right)^2-2\left(h\overline{\nabla}_\mu h+h^{\alpha\beta}\overline{\nabla}_\mu h_{\alpha\beta}\right)$$

$$\frac{\overline{\nabla}^\mu\bar{g}}{\bar{g}}++h^{\mu\alpha}h^\upsilon_\alpha\frac{\overline{\nabla}_\mu\bar{g}}{\bar{g}}\frac{\overline{\nabla}_\upsilon\bar{g}}{\bar{g}}+\frac{\left(\overline{\nabla}\bar{g}\right)^2}{\bar{g}^2}+\kappa\left(2\overline{\nabla}_\mu h\frac{\overline{\nabla}^\mu\bar{g}}{\bar{g}}-h^{\mu\upsilon}\frac{\overline{\nabla}_\mu\bar{g}}{\bar{g}}\frac{\overline{\nabla}_\upsilon\bar{g}}{\bar{g}}\right)\Bigg\}.$$

(100)

The only way the linear term can vanish is by restricting either the allowed fluctuations or else the allowed backgrounds through

线性项为零的唯一途径是通过如下方式限制允许的涨落或限制允许的背景

$$\frac{h}{n}\bar{R}-h^{\alpha\beta}\bar{R}_{\alpha\beta}-\overline{\nabla}^2 h+\overline{\nabla}_\alpha\overline{\nabla}_\beta h^{\alpha\beta}+2\overline{\nabla}_\mu h\frac{\overline{\nabla}^\mu\bar{g}}{\bar{g}}-h^{\mu\upsilon}\frac{\overline{\nabla}_\mu\bar{g}}{\bar{g}}\frac{\overline{\nabla}_\upsilon\bar{g}}{\bar{g}}=0. \tag{101}$$

For maximally symmetric backgrounds, in which

对于最大对称背景，其中

$$\bar{R}_{\mu\nu}=-\frac{2\lambda}{n-2}g_{\mu\nu}, \tag{102}$$

Eq. (101) reads

式 (101) 形式为

$$\overline{\nabla}^2 h-\overline{\nabla}_\alpha\overline{\nabla}_\beta h^{\alpha\beta}+2\overline{\nabla}_\mu h\frac{\overline{\nabla}^\mu\bar{g}}{\bar{g}}-h^{\mu\upsilon}\frac{\overline{\nabla}_\mu\bar{g}}{\bar{g}}\frac{\overline{\nabla}_\upsilon\bar{g}}{\bar{g}}=0. \tag{103}$$

A simple solution is restricting the background to be unimodular, $\bar{g}=1$. Then the offending terms either vanish or else behave as total derivatives. Once a unimodular background is chosen, the linear term corresponds to the EM for the background field

一个简单的解是要求背景为幺模背景，$\bar{g}=1$。此时所有不合要求的项要么等于零，要么表现为全导数。选定幺模背景后，线性项对应背景场的爱因斯坦方程

$$h^{\mu\upsilon}\left(\bar{R}_{\mu\upsilon}-\frac{1}{n}\bar{R}g_{\mu\upsilon}\right)=0. \tag{104}$$

The analysis of the EM of both theories around arbitrary backgrounds $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ can now be done. The Lagrangians for UG and GR with a CC, [11] both expanded up to second order in linear perturbations, are

现在可以对两种理论围绕任意背景 $g_{\mu\nu}$ 和 $\hat{g}_{\mu\nu}$ 做爱因斯坦方程分析。带有宇宙学常数的幺模引力 (UG) 和广义相对论的拉格朗日量 [11] 都按线性微扰展开至二阶，结果为

$$
\mathcal{L}_U = \frac{n+2}{4n^2}\nabla^\mu h\overline{\nabla}_\mu h - \frac{1}{n}\overline{\nabla}_\mu h\overline{\nabla}^\rho h^\mu_\rho + \frac{1}{2}\overline{\nabla}_\mu h^{\mu\rho}\overline{\nabla}_\nu h^\nu_\rho - \frac{1}{4}\overline{\nabla}_\mu h^{\nu\rho}\nabla^\mu h_{\nu\rho} -
$$

$$
-\bar{R}_{\upsilon\beta}h^\beta_\alpha h^{\upsilon\alpha} + \frac{1}{n}h\bar{R}_{\alpha\beta}h^{\alpha\beta} - \frac{\bar{R}}{2}\left(\frac{h^2}{n^2} - \frac{1}{n}h^{\alpha\beta}h_{\alpha\beta}\right), \tag{105}
$$

and

和

$$
\mathcal{L}_{GR\lambda} = \frac{1}{4}\widehat{\nabla}^\mu h\widehat{\nabla}_\mu h - \frac{1}{2}\widehat{\nabla}_\mu h\widehat{\nabla}^\rho h^\mu_\rho + \frac{1}{2}\widehat{\nabla}_\mu h^{\mu\rho}\widehat{\nabla}_\nu h^\nu_\rho - \frac{1}{4}\widehat{\nabla}_\mu h^{\nu\rho}\widehat{\nabla}^\mu h_{\nu\rho} -
$$

$$
-\widehat{R}_{\upsilon\beta}h^\beta_\alpha h^{\upsilon\alpha} + \frac{1}{2}h\widehat{R}_{\alpha\beta}h^{\alpha\beta} - \frac{\widehat{R}+2\lambda}{2}\left(\frac{h^2}{4} - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta}\right). \tag{106}
$$

Assuming both backgrounds to correspond to maximally symmetric spaces in Eq. (102), Eqs. (105) and (106) reduce to

若假设式 (102) 中的两个背景都是最大对称空间，则式 (105) 和 (106) 简化为

$$
\mathcal{L}_U = \frac{n+2}{4n^2}\nabla^\mu h\overline{\nabla}_\mu h - \frac{1}{n}\overline{\nabla}_\mu h\overline{\nabla}^\rho h^\mu_\rho + \frac{1}{2}\overline{\nabla}_\mu h^{\mu\rho}\overline{\nabla}_\nu h^\nu_\rho - \frac{1}{4}\overline{\nabla}_\mu h^{\nu\rho}\nabla^\mu h_{\nu\rho},
$$

(107)

$$
\mathcal{L}_{GR\lambda} = \frac{1}{4}\widehat{\nabla}^\mu h\widehat{\nabla}_\mu h - \frac{1}{2}\widehat{\nabla}_\mu h\widehat{\nabla}^\rho h^\mu_\rho + \frac{1}{2}\widehat{\nabla}_\mu h^{\mu\rho}\widehat{\nabla}_\nu h^\nu_\rho - \frac{1}{4}\widehat{\nabla}_\mu h^{\nu\rho}\widehat{\nabla}^\mu h_{\nu\rho} -
$$

$$
-\frac{\lambda}{2}\left(\frac{h^2}{2} - h_{\alpha\beta}h^{\alpha\beta}\right). \tag{108}
$$

Sources for both theories can be introduced in the usual way by a linear coupling. In the case of GR, the source is just the usual symmetric energy-momentum tensor $T_{\mu\nu}$, while UG, as discussed above, couples to the traceless source $J_{\mu\nu}$, which obeys $\nabla_\mu J^{\mu\nu} = \nabla^\nu T$

两种理论都可以通过线性耦合按常规方式引入源。广义相对论中，源就是常规的对称能量动量张量 $T_{\mu\nu}$ ，而如上所述，幺模引力与无迹源 $J_{\mu\nu}$ 耦合，该源满足 $\nabla_\mu J^{\mu\nu} = \nabla^\nu T$

The EM for UG, dubbed EMU, read

幺模引力的爱因斯坦方程记为 EMU，形式为

$$
EMU \equiv \frac{n+2}{2n^2}g_{\mu\upsilon}\overline{\nabla}^2 h - \frac{1}{2}\overline{\nabla}^2 h_{\mu\upsilon} - \frac{1}{n}\overline{\nabla}_\alpha\overline{\nabla}_\beta h^{\alpha\beta}g_{\mu\upsilon} -
$$

$$
\frac{1}{n}\overline{\nabla}_\mu\overline{\nabla}_\nu h + \frac{1}{2}\overline{\nabla}_\mu\overline{\nabla}_\alpha h^\alpha_\upsilon + \frac{1}{2}\overline{\nabla}_\nu\overline{\nabla}_\alpha h^\alpha_\mu = J_{\mu\nu} \tag{109}
$$

whereas the EM of general relativity, EMGR, are

而广义相对论的爱因斯坦方程记为 EMGR，形式为

$$EMGR \equiv \frac{1}{2}\hat{\nabla}^2 h\hat{g}_{\mu\upsilon} - \frac{1}{2}\hat{\nabla}^2 h_{\mu\upsilon} - \frac{1}{2}\hat{\nabla}_\alpha\hat{\nabla}_\beta h^{\alpha\beta}\hat{g}_{\mu\upsilon} - \frac{1}{2}\hat{\nabla}_\mu\hat{\nabla}_\upsilon h +$$

$$\frac{1}{2}\hat{\nabla}_\mu\hat{\nabla}_\alpha h^\alpha_\upsilon + \frac{1}{2}\hat{\nabla}_\upsilon\hat{\nabla}_\alpha h^\alpha_\mu = \lambda\left(\frac{h}{2}\hat{g}_{\mu\upsilon} - h_{\mu\upsilon}\right) + T_{\mu\upsilon}. \tag{110}$$

---

[11] In the full non-linear theory, the CC is included in an arbitrary energy-momentum tensor. In the linear approximation, this is not the case.

[11] 在完整非线性理论中，宇宙学常数被包含在任意能量动量张量中，但在线性近似中并非如此。

At this point, the result advertised in the introduction on the equivalence of the unimodular theory with GR with an undetermined CC should be remembered. Fluctuations around a flat background were already analyzed in the previous section, and full equivalence with GR with vanishing CC was found. To ensure that this result is not an artifact of the flat background, it is worth repeating the analysis in this more general setting.

到此, 我们应当还记得引言中给出的结论: 幺模引力理论等价于宇宙学常数未确定的广义相对论。上一节已经分析了平直背景下的涨落，发现它确实等价于宇宙学常数为零的广义相对论。为了确认这个结果不是平直背景带来的人为产物，我们有必要在这个更一般的框架下重新分析。

The first integrals from the EM can be derived to make things easy. The first one, I_{GR} , is obtained by taking the trace of the EMGR

为简化计算，我们可以推导爱因斯坦方程给出的第一积分。对广义相对论的爱因斯坦方程求迹即可得到第一个第一积分 I_{GR} ，

$$I_{GR} \equiv \hat{\nabla}^2 h - \hat{\nabla}_\alpha\hat{\nabla}_\beta h^{\alpha\beta} - \lambda h = \frac{2}{n-2}T, \tag{111}$$

whereas the second one stems from taking the covariant divergence of the EMU

而第二项来源于对 EMU 求协变散度

$$I_U \equiv \frac{2-n}{2n}\overline{\nabla}_\upsilon\left(\overline{\nabla}_\alpha\overline{\nabla}_\beta h^{\alpha\beta} - \frac{1}{n}\overline{\nabla}^2 h + \frac{1}{n}T\right) = 0$$

$$\Rightarrow \overline{\nabla}_\alpha\overline{\nabla}_\beta h^{\alpha\beta} - \frac{1}{n}\overline{\nabla}^2 h + \frac{1}{n}T = \Gamma, \tag{112}$$

where Γ is an arbitrary constant. Assume that the background metric is the same for both theories, [12] $g_{\mu\nu} = \hat{g}_{\mu\nu}$. After that, perform a field redefinition of the form $h_{\mu\nu} = H_{\mu\nu} + aHg_{\mu\nu}$ that would map one theory into the other, with the possible addition of terms proportional to the first integrals, which are zero by the use of the EM. This is equivalent to a search for constants a, C_1, C_2 , and Γ such that

其中Γ为任意常数。假设两个理论的背景度规相同,[12]$g_{\mu\nu}=\hat{g}_{\mu\nu}$。之后,进行形式为 $h_{\mu\nu}=H_{\mu\nu}+aHg_{\mu\nu}$ 的场重新定义，将一个理论映射到另一个，可能会添加正比于第一积分的项，这些项利用 EM 可证为零。这等价于寻找常数 a, C_1, C_2 和 Γ 使得

$$EMGR\left(H_{\mu\nu}+aHg_{\mu\nu}\right)+C_2 I_{GR}\left(H_{\mu\nu}+aHg_{\mu\nu}\right)=$$

$$=EMU\left(H_{\mu\nu}\right)+C_1 I_U\left(H_{\mu\nu}\right), \tag{113}$$

and

且

$$\overline{\nabla}^2 Hg_{\mu\nu}\left(\frac{n+2}{2n^2}-\frac{C_1}{n}-\frac{1}{2}-a\left(\frac{n}{2}-1\right)-C_2\left(1+na-a\right)\right)+$$

$$+\overline{\nabla}_\alpha\overline{\nabla}_\beta H^{\alpha\beta}g_{\mu\nu}\left(C_1+C_2+\frac{1}{2}-\frac{1}{n}\right)+\overline{\nabla}_\mu\overline{\nabla}_\nu H\left(\frac{1}{2}-\frac{1}{n}-a\left(1-\frac{n}{2}\right)\right)+$$

$$+Hg_{\mu\nu}\lambda\left(\frac{1}{2}+a\left(\frac{n}{2}-1\right)+C_2\left(1+na\right)\right)+g_{\mu\nu}\left(T\left(\frac{2C_2}{n-2}+\frac{C_1+1}{n}\right)-C_1\Gamma\right)+$$

$$+T_{\mu\nu}-J_{\mu\nu}-\lambda H_{\mu\nu}=0. \tag{114}$$

---

[12] This is a reasonable ansatz to check equivalence.

[12] 这是一个检验等价性的合理拟设。

The equations obtained by demanding every factor to be zero are only compatible if the CC λ vanishes. In that case, the solution of the system is simply

要求所有系数都为零得到的方程组仅在宇宙学常数 λ 为零时相容，此时方程组的解可简单写为

$$a=-\frac{1}{n}$$

$$C_1+C_2=\frac{2-n}{2n}$$

$$\Gamma=\left(\frac{n^2\left(2C_2-1\right)+n\left(4C_2+4\right)-4}{2n^2\left(n-2\right)}\right)T. \tag{115}$$

To summarize,

综上所述

$$EMGR\left(H_{\mu\upsilon}-\frac{1}{n}Hg_{\mu\upsilon}\right)-\frac{n-2}{2n}I_{GR}\left(H_{\mu\upsilon}-\frac{1}{n}Hg_{\mu\upsilon}\right)=$$

$$=EMU\left(H_{\mu\upsilon}\right)\Big|_{J_{\mu\upsilon}=T_{\mu\upsilon}-\frac{1}{n}Tg_{\mu\upsilon}}. \tag{116}$$

The physical meaning of what has been proved is that the unimodular EMU are a consequence of EMGR when $\lambda=0$ only; it is the subsector corresponding to

上述结论的物理意义是: 仅当 $\lambda=0$ 时，幺模 EMU 是 EMGR 的自然结果；它对应的子分支是

$$h_{\mu\upsilon}^{GR}=h_{\mu\upsilon}^{U}-\frac{1}{n}h^{U}g_{\mu\upsilon}. \tag{117}$$

It is worth remarking (again) that this is not a field redefinition; it is a truncation of GR such that $h^{GR}=0$. There is no way to build the inverse map from EMU to EMGR. Given the fact that

值得再次指出，这并不是场重新定义；它是广义相对论满足 $h^{GR}=0$ 的一种截断。无法构建从 EMU 到 EMGR 的逆映射。鉴于

$$h=0, \tag{118}$$

is a (partial) algebraic gauge fixing (which does not need ghosts); this shows that, at the level of the EM, the unimodular theory is a truncation of GR with vanishing CC and with the source reduced to the traceless part of the GR source. It is perhaps worth remarking that this does not necessarily follow from the fact that the Lagrangian is so obtained (gauge conditions can only be used after the EM are derived).

是 (部分) 代数规范固定 (不需要鬼场)；这表明，在运动方程层面，幺模理论是广义相对论在宇宙学常数为零、源约化为广义相对论源无迹部分后的一种截断。或许需要指出，这一点并不必然能从拉格朗日量的构造方式直接得到 (规范条件只能在运动方程导出后使用)。

No Birkhoff's Theorem in UG

幺模引力中不存在伯克霍夫定理

It would seem that Birkhoff's theorem trivially holds in UG. After all, it is claimed [12] that UG is (classically at least) equivalent to GR with a CC unrelated to vacuum energy. So this would seem to close the argument.

乍看之下伯克霍夫定理在幺模引力中显然成立。毕竟已有文献 [12] 声称，(至少经典层面上) 幺模引力等价于带宇宙学常数的广义相对论，且该宇宙学常数与真空能无关。因此这个问题似乎已经有了结论。

Another urban myth is that Birkhoff's theorem is related to the fact that GR only propagates spin two without any spin zero impurities. Given that UG also propagates spin two only, it should follow that Birkhoff's theorem should also hold for UG.

另一个常见误区是，认为伯克霍夫定理与广义相对论只传播自旋 2 模式、不含自旋 0 杂质这一事实相关。既然幺模引力也只传播自旋 2 模式，顺理成章伯克霍夫定理也应当在幺模引力中成立。

Things are not so simple; recall the vacuum EM for UG derived in section ”Nonlinear Field Theory”:

事实并没有这么简单；我们来回顾一下在“非线性场论”一节中推导得到的幺模引力真空场方程:

$$R_{\mu\upsilon} - \frac{1}{n}Rg_{\mu\upsilon} + \frac{(2-n)(2n-1)}{4n^2}\left(\frac{\nabla_\mu g\nabla_\upsilon g}{g^2} - \frac{1}{n}\frac{(\nabla g)^2}{g^2}g_{\mu\upsilon}\right) + \\ + \frac{n-2}{2n}\left(\frac{\nabla_\mu\nabla_\upsilon g}{g} - \frac{1}{n}\frac{\nabla^2 g}{g}g_{\mu\upsilon}\right) = 0. \tag{119}$$

In the Weyl gauge $g = 1$, they reduce to

在魏尔规范 $g = 1$ 中，它们可简化为

$$R_{\mu\upsilon} = \frac{1}{n}Rg_{\mu\upsilon}. \tag{120}$$

The way to solve Eq. (119) is first to solve Eq. (120) for some unimodular metric $\gamma_{\alpha\beta}$. Then any Weyl rescaling of $\gamma_{\alpha\beta}$ is a solution to Eq. (119). The EM imply that in vacuum UG, spacetimes in this Weyl gauge are constant curvature spaces. Therefore, a detailed analysis of the collapse should be made to ascertain under what initial physical conditions a nonvanishing curvature is generated. Besides, many physical properties are not Weyl invariant. For example, flat space is conformally related to both de Sitter and anti-de Sitter spacetimes, both being Petrov type 0.

求解方程 (119) 的方法是先针对某个幺模度规 $\gamma_{\alpha\beta}$ 求解方程 (120)，那么 $\gamma_{\alpha\beta}$ 的任意魏尔缩放都是方程 (119) 的解。场方程表明，在真空幺模引力中，该魏尔规范下的时空都是常曲率空间。因此，我们需要对坍缩过程开展详细分析，以确定在何种初始物理条件下会生成非零曲率。此外，很多物理性质不具有魏尔不变性。例如，平坦空间可共形联系到德西特和反德西特时空，二者都是佩特罗夫 0 型。

All solutions to the complete UG EM can be characterized by one unimodular solution $\gamma_{\alpha\beta}$ in the Weyl gauge $g = 1$ together with all their Weyl rescalings, $g_{\alpha\beta} = g^{\frac{1}{4}}\gamma_{\alpha\beta}$. The unimodular metric γ is equivalent to the selection of a particular set of eigenvalues for one of the Petrov types (this being a Weyl invariant classification); see Fig. 1.

完整幺模引力场方程的所有解都可以用以下方式刻画: 魏尔规范 $g = 1$ 下的一个幺模解 $\gamma_{\alpha\beta}$ ，加上该解的所有魏尔缩放 $g_{\alpha\beta} = g^{\frac{1}{4}}\gamma_{\alpha\beta}$ 。幺模度规 γ 等价于为某类佩特罗夫型选定一组特定特征值 (佩特罗夫分类是魏尔不变的)；参见图 1。

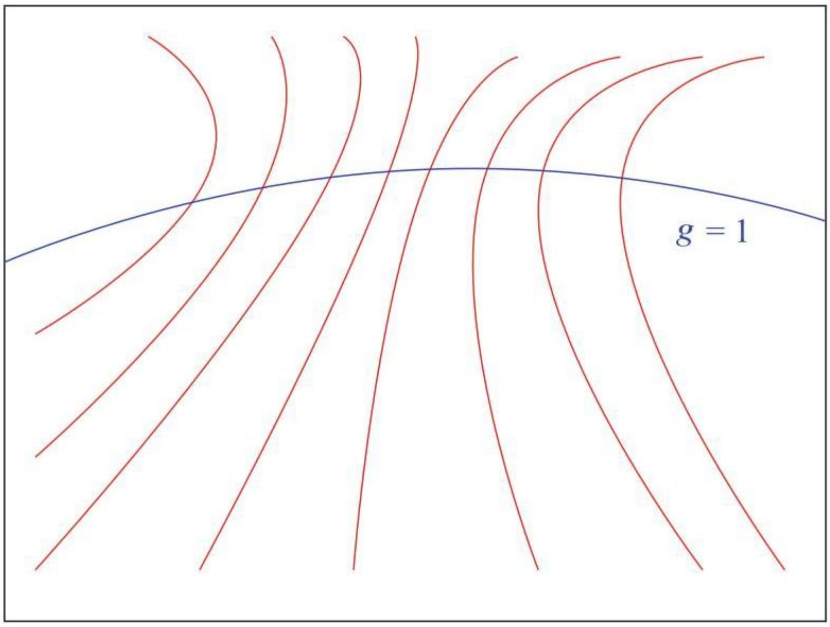


Fig. 1 Weyl gauge orbits and gauge fixing for metrics $g = 1$

图 1 度规 $g = 1$ 的魏尔规范轨道与规范固定

Unimodular Cosmology

幺模宇宙学

Consider the simplest cosmological model keeping the topology of the constant time surfaces flat. The corresponding metric in the unimodular gauge, $g = 1$, reads

考虑保持等时曲面拓扑为平坦的最简单宇宙学模型。幺模规范下的对应度规 $g = 1$ 形式如下

$$ds^2 = b(t)^{-3/2} dt^2 - b(t)^{1/2} \delta_{ij} dx^i dx^j, \tag{121}$$

where $b = b(t)$, only depends on time. The cosmic normalized four-velocity vector field, $u^\mu u_\mu = 1$, is given explicitly by

其中 $b = b(t)$ 仅依赖于时间。宇宙归一化四维速度矢量场 $u^\mu u_\mu = 1$ 可显式表示为

$$u^\mu = \left(b^{3/4}, 0, 0, 0\right). \tag{122}$$

and the projector onto the three-space is

到三维空间的投影算子为

$$\Pi^{\mu}_{\upsilon} \equiv \delta^{\mu}_{\upsilon} - u^{\mu} u_{\upsilon} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{123}$$

It is easy to check that this congruence is geodesic

不难验证该同余是测地线

$$u^{\upsilon} \nabla_{\upsilon} u^{\mu} = 0, \tag{124}$$

and the volume expansion reads

体积膨胀可写为

$$\theta \equiv \nabla_{\mu} u^{\mu} = \frac{3}{4} b^{-1/4} \frac{db}{dt}. \tag{125}$$

The unimodular gauge has been used extensively, especially at the dawn of GR, in particular by Einstein himself and also by Schwarzschild [22].

幺模规范已被广泛使用，尤其在广义相对论发展初期，爱因斯坦本人以及史瓦西都使用过该规范 [22]。

Once again, for this gauge, the EM in UG correspond to

再次说明，对于该规范，幺模引力中的场方程对应为

$$R_{\mu\upsilon} - \frac{1}{4} R g_{\mu\upsilon} = 2\kappa^2 \left(T_{\mu\upsilon} - \frac{1}{4} T g_{\mu\upsilon} \right), \tag{126}$$

where $\kappa^2 \equiv 8\pi G$. The scalar curvature corresponding to Eq. (121) is

其中 $\kappa^2 \equiv 8\pi G$ 。式 (121) 对应的标量曲率为

$$R = -\frac{3}{8\sqrt{b}} \left[\left(\frac{db}{dt} \right)^2 + 4b \frac{d^2 b}{dt^2} \right]. \tag{127}$$

Considering matter as a perfect fluid with a stress-energy tensor,

将物质视为具有能量动量张量的理想流体，

$$T_{\mu\upsilon} \equiv (\rho + p)\, u_{\mu} u_{\upsilon} - p g_{\mu\upsilon}, \tag{128}$$

for which the energy-momentum conservation, $\nabla_{\upsilon} T^{\mu\upsilon} = 0$, is then equivalent to

此时能量动量守恒 $\nabla_{\upsilon} T^{\mu\upsilon} = 0$ 等价于

$$u^{\mu}\nabla_{\mu}\rho+(\rho+p)\,\theta=0. \tag{129}$$

Using Eq. (126) along with Eq. (124), Raychaudhuri's equation [23] reduces to

结合式 (126) 与式 (124)，瑞乔杜里方程 [23] 可简化为

$$u^{\mu}\nabla_{\mu}\theta+\frac{1}{n-1}\theta^{2}+\sigma_{\alpha\beta}\sigma^{\alpha\beta}-\omega_{\alpha\beta}\omega^{\alpha\beta}+\frac{1}{n}R+\frac{2\,(n-1)}{n}\kappa^{2}\,(\rho+p)=0. \tag{130}$$

Using the explicit form for the Ricci scalar in (127),

利用 (127) 中里奇标量的显式形式,

$$R=-2u^{\mu}\nabla_{\mu}\theta-\frac{4}{3}\theta^{2}. \tag{131}$$

Assuming, as it is usual for simplicity, vanishing shear and rotation, $\sigma_{\alpha\beta}=\omega_{\alpha\beta}=0$, in the physical dimension $n=4$,

如通常为简化所做的假设，物理维度 $n=4$ 中切变和旋转为零 $\sigma_{\alpha\beta}=\omega_{\alpha\beta}=0$,

$$u^{\mu}\nabla_{\mu}\theta+3\kappa^{2}\,(\rho+p)=0. \tag{132}$$

It is worth remarking that, due to the tracelessness of the EM, it is not possible to express R in terms of T .

值得注意的是，由于场方程无迹，无法用 T 表示 R 是不可能的。

Using Ellis' clever trick [24], one can define a length scale through

利用埃利斯的巧妙技巧 [24]，我们可以通过下式定义长度尺度

$$\theta=\frac{3}{l}u^{\mu}\nabla_{\mu}l \tag{133}$$

actually

实际上

$$b\sim l^{4}\,. \tag{134}$$

Finally, one can write Raychaudhuri's equation (132) like

最后，我们可以将瑞乔杜里方程 (132) 写为

$$u^{\mu}u^{\upsilon}\left[l\nabla_{\upsilon}\nabla_{\mu}l-\nabla_{\mu}l\nabla_{\upsilon}l\right]+\kappa^{2}\,(\rho+p)\,l^{2}=0. \tag{135}$$

In vacuum, $p = \rho = 0$, and Raychaudhuri' s equation reduces to

在真空中，$p = \rho = 0$ ，此时瑞乔杜里方程简化为

$$u^{\mu}\nabla_{\mu}\theta = 0. \tag{136}$$

This is easy to calculate:

计算过程很简单:

$$u^{\mu}\nabla_{\mu}\theta = u^{\upsilon}\nabla_{\upsilon}\nabla_{\mu}u^{\mu} = -\frac{3}{16\sqrt{b}}\left[\left(\frac{db}{dt}\right)^{2} - 4b\frac{d^{2}b}{dt^{2}}\right] = 0. \tag{137}$$

The vacuum EM for UG read

幺模引力的真空场方程为

$$\left(\frac{db}{dt}\right)^{2} - 4b\frac{d^{2}b}{dt^{2}} = 0. \tag{138}$$

The general solution to Eq. (138) is given by either

方程 (138) 的通解可以是

$$b = b_0 \tag{139}$$

(a constant) corresponding to flat space, or else by

(一个常数) 对应平直空间，或是

$$b\,(t) = H_0^{\frac{4}{3}}(3t - t_0)^{\frac{4}{3}}, \tag{140}$$

which corresponds to de Sitter space. [13] Here, $H_0 \equiv 3\theta$ is the (constant) expansion rate, which for this solution is arbitrary, as there is no physical scale in the problem that can determine it. It is to be emphasized that this solution depends on two parameters, whereas the flat space solution depends only on one, being thus less generic.

对应德西特空间。[13] 此处，$H_0 \equiv 3\theta$ 是 (常数) 膨胀率，该解中膨胀率是任意的，因为本问题中不存在能确定它的物理标度。需要强调的是，这个解依赖两个参数，而平直空间解仅依赖一个参数，因此普适性更低。

This could have been anticipated because the vacuum EM in UG are just Einstein spaces:

这一点是可以预见的，因为幺模引力的真空场方程恰好就是爱因斯坦空间方程:

$$R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}. \tag{143}$$

Flat space is just a pretty particular solution; non-zero constant curvature spacetimes [25] constitute the more generic ones.

平直空间只是一个非常特殊的解；非零常数曲率时空 [25] 才是更具普适性的解。

It is sometimes asserted that UG is equivalent to general relativity in the gauge $g = 1$. However, this is not true, as shown in this paragraph.

时常有观点认为幺模引力在规范 $g = 1$ 下等价于广义相对论，但正如本段所示，这个说法并不正确。

First, the synchronous gauge cannot be reached with the residual gauge symmetry once in the unimodular gauge. The best one can do [26] is

首先，进入幺模规范后，剩余规范对称性无法得到同步规范。现有研究 [26] 能得到的最佳结果是

$$ds^2 = a(t)\, dt^2 - R^2(t)\, \delta_{ij} dx^i dx^j, \tag{144}$$

where flat spatial sections have been chosen for simplicity. Einstein's equations with vacuum energy

为简化起见，这里选择了平直空间切片。带真空能的爱因斯坦方程

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \; T_{\mu\nu} \equiv \rho g_{\mu\nu}, \tag{145}$$

---

13 In unimodular coordinates, the maximally symmetric, constant curvature de Sitter spacetime reads

13 在幺模坐标中，最大对称、常曲率德西特时空可写为

$$ds^2 = \left(\frac{dt}{3Ht}\right)^2 - (3Ht)^{2/3} \delta_{ij} dx^i dx^j \tag{141}$$

that is, precisely

也就是说，恰好是

$$b(t) \sim t^{\frac{4}{3}} \tag{142}$$

---

read

为

$$G_{00} = 3\frac{\dot{R}^2}{R^2} = \kappa^2 a \rho,$$

$$G_{ij} = \frac{\dot{a}\dot{R}R - 2a\ddot{R}R - a\dot{R}^2}{a^2}\delta_{ij} = -\kappa^2\rho R^2\delta_{ij}. \tag{146}$$

The unimodular gauge implies

幺规范意味着

$$aR^6 = 1\,, \tag{147}$$

for which the straightforward solution is

其直接解为

$$R = \left[R_0^3 + \kappa\sqrt{\frac{\rho}{3}}\,(t - t_0)\right]^{\frac{1}{3}},$$

$$a = \left[R_0^3 + \kappa\sqrt{\frac{\rho}{3}}\,(t - t_0)\right]^{-2}. \tag{148}$$

This is just flat space in vacuum ($\rho = 0$) and an exponential expansion in the synchronous time

这就是真空中的平直空间 ($\rho = 0$)，对应共动时间下的指数膨胀

$$\tau = \int^t \frac{dx}{R_0^3 + \kappa\sqrt{\frac{\rho}{3}}\,(x - t_0)} = \frac{1}{\kappa}\sqrt{\frac{3}{\rho}}\log\left(R_0^3 + \kappa\sqrt{\frac{\rho}{3}}\,(t - t_0)\right), \tag{149}$$

in such a way that

满足如下关系

$$R\,(\tau) = e^{\frac{\kappa}{3}\sqrt{\frac{\rho}{3}}(\tau - \tau_0)}, \tag{150}$$

and the exponential expansion disappears once $\rho = 0$.

当 $\rho = 0$ 时，指数膨胀消失

On the other hand, the unimodular gauge of GR is, of course, fully equivalent to the usual formulation of GR in comoving coordinates [27,28] where the metric reads

另一方面，广义相对论的幺规范当然完全等价于共动坐标下的广义相对论通常表述 [27,28]，其中度规可写为

$$ds^2 = d\tau^2 - a(\tau)^2 \sum \delta_{ij} dx^i dx^j, \tag{151}$$

with a four-velocity

其中四速度为

$$u^{\mu} = (1, 0, 0, 0) \tag{152}$$

and

且

$$u^{\nu}\nabla_{\nu}u^{\mu} = 0. \tag{153}$$

In this case,

在这种情况下

$$\theta = 3\frac{1}{a}\frac{da}{dt}. \tag{154}$$

Again, the only difference between GR and UG stems from the EM. In detail, now the EM are the usual Einstein ones

由此可见，广义相对论与幺模引力的唯一差异同样来自场方程。具体来说，此处的场方程就是常规的爱因斯坦场方程

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\kappa^2 T_{\mu\nu}, \tag{155}$$

in this case, the scalar of curvature reads

此时曲率标量可写为

$$R = -\frac{6}{a^2}\left[\left(\frac{da}{dt}\right)^2 + a\frac{d^2a}{dt^2}\right]. \tag{156}$$

Raychaudhuri's equation in comoving coordinates yields

共动坐标下的雷乔杜里方程给出

$$-3\left(\frac{da}{dt}\right)^2 + 2a^2\kappa^2\rho = 0. \tag{157}$$

In this case, the vacuum solution reduces to

这种情况下，真空解退化为

$$\frac{da}{dt} = 0 \tag{158}$$

i.e., $\theta = 0$, which is just flat spacetime. This is a subset of the unimodular result $\dot{\theta} = 0$.

即 $\theta = 0$，这就是平直时空。它是幺模结果 $\dot{\theta} = 0$. 的一个子集

Tree Diagrams

树图

A natural question to ask at this stage is whether the S-matrix would be the same for UG as for GR. The propagators and the vertices are pretty different in both theories, so the answer to this question is not immediate. In [29], the calculation of the nonvanishing tree-level three, four, and five graviton amplitudes were compared with the diagrams in GR [30-32]. Complete agreement at the diagram level was found. In this section, the key ingredients for that computation are presented.

在现阶段自然会提出一个问题: 幺正引力 (UG) 的 S 矩阵是否与广义相对论 (GR) 的相同。两种理论中的传播子和顶角都截然不同，因此这个问题无法立刻得到答案。在文献 [29] 中，作者对非零树 level 的 3、4、5 引力子振幅计算结果与广义相对论中的图结果 [30-32] 进行了比较，发现在图层面结果完全一致。本节将介绍该计算的核心要素。

To obtain the Feynman rules for UG, consider the action Eq. (31). The propagator is obtained by inverting the second-order expansion [14] of the Lagrangian - once properly gauge-fixed - presented in [21]. This reads

为推导幺正引力的费曼规则，我们考察式 (31) 的作用量。如文献 [21] 所示，经过恰当规范固定后，传播子可以通过对拉格朗日量的二阶展开 [14] 求逆得到，表达式为:

$$\mathcal{L} = \frac{1}{4}h^{\mu\upsilon}\partial^2 h_{\mu\upsilon} - \frac{1}{4n}h\partial^2 h + \left(-f\partial^2 f + \frac{\alpha}{2}f\partial^2 h + \frac{\alpha}{2}h\partial^2 f\right) -$$
$$-\frac{1}{2}\left(\partial_\mu c'^{(0,0)}\partial^\mu c'^{(0,0)} + 2\left(\partial_\upsilon h^\upsilon_\mu - \frac{1}{n}\partial_\mu h\right)\partial^\mu c'^{(0,0)}\right) \tag{159}$$

---

[14] Around a flat background, $\partial^2 \equiv \partial_\mu\partial_\upsilon\eta_{\mu\upsilon}$.

[14] 在平直背景附近，$\partial^2 \equiv \partial_\mu\partial_\upsilon\eta_{\mu\upsilon}$ 。

The action can be written as

作用量可以写为

$$S = \int d^n x \Psi^A F_{AB} \Psi^B, \tag{160}$$

where

其中

$$F_{AB} = G_{AB}\partial^2 + J_{AB}^{\mu\upsilon}\partial_\mu\partial_\upsilon, \tag{161}$$

and

且

$$\Psi^A = \begin{pmatrix} h^{\mu\upsilon} \\ f \\ c' \end{pmatrix}. \tag{162}$$

The auxiliary tensors are defined as

辅助张量定义为

$$\mathcal{P}^{\alpha\beta}_{\mu\upsilon\rho\sigma} = \frac{1}{4}\left(\eta_{\mu\rho}\delta_\upsilon^{(\alpha}\delta_\sigma^{\beta)} + \eta_{\mu\sigma}\delta_\upsilon^{(\alpha}\delta_\rho^{\beta)} + \eta_{\upsilon\rho}\delta_\mu^{(\alpha}\delta_\sigma^{\beta)} + \eta_{\upsilon\sigma}\delta_\mu^{(\alpha}\delta_\rho^{\beta)}\right), \tag{163}$$

$$\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\eta_{\mu\nu}\delta_\rho^{(\alpha}\delta_\sigma^{\beta)} + \eta_{\rho\sigma}\delta_\mu^{(\alpha}\delta_\nu^{\beta)}\right). \tag{164}$$

The different matrices involved read

涉及的各矩阵如下:

$$G_{AB} = \begin{pmatrix} -\frac{1}{4}\left(\frac{1}{4}\mathcal{K}^{\alpha\beta}_{\mu\nu\rho\sigma} - \mathcal{P}^{\alpha\beta}_{\mu\nu\rho\sigma}\right)\eta_{\alpha\beta} & \frac{\alpha}{2}\eta_{\mu\upsilon} & -\frac{1}{8}\eta_{\mu\upsilon} \\ \frac{\alpha}{2}\eta_{\rho\sigma} & -1 & 0 \\ -\frac{1}{8}\eta_{\rho\sigma} & 0 & \frac{1}{2} \end{pmatrix}, \tag{165}$$

$$J^{\alpha\beta}_{AB} = \begin{pmatrix} 0 & 0 & \frac{1}{4}\left(\delta_\mu^\alpha\delta_\nu^\beta + \delta_\nu^\alpha\delta_\mu^\beta\right) \\ 0 & 0 & 0 \\ \frac{1}{4}\left(\delta_\rho^\alpha\delta_\sigma^\beta + \delta_\sigma^\alpha\delta_\rho^\beta\right) & 0 & 0 \end{pmatrix}. \tag{166}$$

For the gauge choice of [21], the graviton propagator in UG reads

采用文献 [21] 的规范选择，幺正引力中的引力子传播子为

$$\begin{aligned} P^{\mathrm{UG}}_{\mu\nu,\rho\sigma} = {} & \frac{1}{2k^2}\left(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma}\right) - \frac{1}{k^2}\frac{\alpha^2 n^2 - n + 2}{\alpha^2 n^2 (n-2)}\eta_{\mu\nu}\eta_{\rho\sigma} + \\ & + \frac{2}{n-2}\left(\frac{k_\rho k_\sigma \eta_{\mu\nu}}{k^4} + \frac{k_\mu k_\nu \eta_{\rho\sigma}}{k^4}\right) - \frac{2n}{n-2}\frac{k_\mu k_\nu k_\rho k_\sigma}{k^6}. \end{aligned} \tag{167}$$

Recall that the usual GR graviton propagator in the de Donder gauge,

回顾一下，通常广义相对论中德唐德规范下的引力子传播子

$$P^{\mathrm{GR}}_{\mu\upsilon\rho\sigma} = \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\upsilon\rho} + \eta_{\mu\rho}\eta_{\upsilon\sigma} - \frac{2}{n-2}\eta_{\mu\upsilon}\eta_{\rho\sigma}\right), \tag{168}$$

has only simple poles at $k^2 = 0$. In the UG propagator, there appear additional double and triple poles. This is a technical complication that impedes a priori the application of the techniques [33] to reduce the computation of the diagrams. Furthermore, it is possible to show that no gauge choice in UG can yield a propagator of the form

仅在 $k^2 = 0$ 处存在单极点。在幺正引力的传播子中，额外出现了双极点和三极点。这是一个技术上的难点，先验地阻碍了 [33] 中化简图计算的技术的应用。此外，可以证明，在幺正引力中不存在能给出如下形式传播子的规范选择

$$P_{\mu\upsilon\rho\sigma} = \frac{i}{2k^2}\left(\eta_{\mu\sigma}\eta_{\upsilon\rho} + \eta_{\mu\rho}\eta_{\upsilon\sigma} - f_1\left(k^2\right)\eta_{\mu\upsilon}\eta_{\rho\sigma}\right) +$$

$$+f_2\left(k^2\right)\left(k_\rho k_\rho \eta_{\mu\upsilon} + k_\mu k_\upsilon \eta_{\rho\rho}\right) + f_3\left(k^2\right)k_\mu k_\upsilon k_\rho k_\rho, \tag{169}$$

$f_3\left(k^2\right)$ having no pole at $k^2 = 0$ if the Newtonian potential is to be obtained in the non-relativistic limit. The triple pole term in (167) is needed to recover the said potential.

$f_3\left(k^2\right)$ 若要在非相对论极限下得到牛顿势，就不能在 $k^2 = 0$ 处留极点，(167) 中的三极点项是恢复该势的必要项。

The three-and-four graviton vertices are needed to calculate the four- and five-point amplitudes. These are obtained from the second- and third-order expansion of the Lagrangian. The results can be presented in a compact form via the parameter n . For GR vertices, $n = 2$, while for UG, $n = 4$. Using the all incoming momenta convention in the diagrams in Figs. 2, 3, and 4, the three-graviton vertex reads

计算四点和五点振幅需要用到三引力子顶角和四引力子顶角，它们由拉格朗日量的三阶展开和四阶展开得到。结果可以通过参数 n 以紧致形式表示。对于广义相对论的顶角，$n = 2$ ，而对于幺正引力，$n = 4$ 。采用图 2、3、4 中所有动量均为入射的约定，三引力子顶角为:

$$V^{\mu\nu,\rho\rho,\alpha\beta}_{(p1,p2,p3)} =$$

$$= i2\kappa\mathcal{S}\left\{\frac{(2+n)\left(p_1\cdot p_2\right)}{2n}\left[\frac{\eta^{\alpha\beta}\eta^{\mu\upsilon}\eta^{\rho\rho}}{n^2} - \frac{2\eta^{\alpha\rho}\eta^{\beta\rho}\eta^{\mu\upsilon}}{n} - \frac{\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\upsilon\rho}}{2+n}\right] +\right.$$

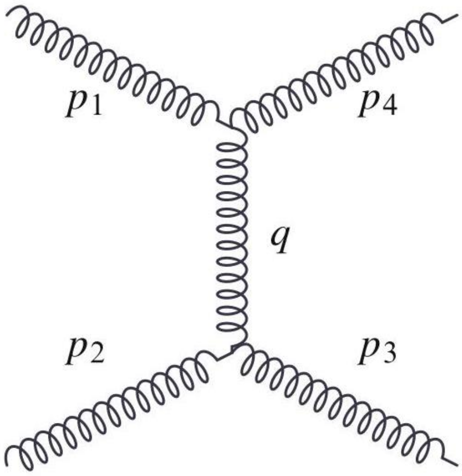
$$+\frac{2\eta^{\beta\nu}\eta^{\rho\rho}p_1^m p_2^\alpha}{n} + \frac{1}{2}\eta^{mr}\eta^{\nu\rho}p_1^\alpha p_2^\beta - \frac{(2+n)\eta^{\mu\nu}\eta^{\rho\rho}p_1^\alpha p_2^\beta}{2n^2} - 2\eta^{\beta\rho}\eta^{\nu\rho}p_1^\alpha p_2^\mu$$

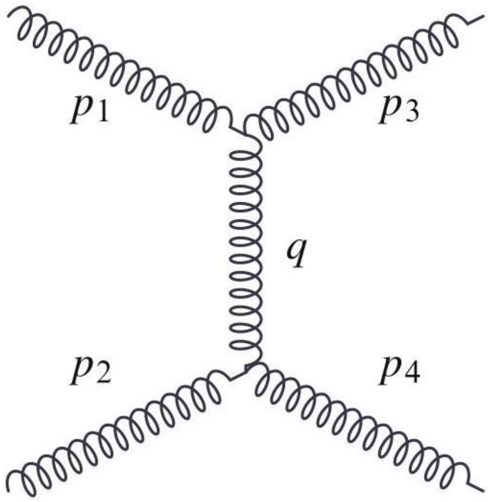
$$-\eta^{\alpha\upsilon}\eta^{\beta\rho}p_1^\rho p_2^\mu + \frac{\eta^{\alpha\beta}\eta^{\upsilon\rho}p_1^\rho p_2^\mu}{n} + \frac{2\eta^{\beta\mu}\eta^{\rho\rho}p_1^\alpha p_2^\upsilon}{n} - \frac{2\eta^{\alpha\beta}\eta^{\rho\rho}p_1^\mu p_2^\upsilon}{n^2} +$$

$$\left. + \frac{2\eta^{\alpha\mu}\eta^{\beta\upsilon}p_1^\rho p_2^\rho}{\upsilon} + \left(p_1\cdot p_2\right)\eta^{\alpha\upsilon}\eta^{\beta\rho}\eta^{\mu\rho}\right\}. \tag{170}$$

Fig. 2u channel

图 2u 道





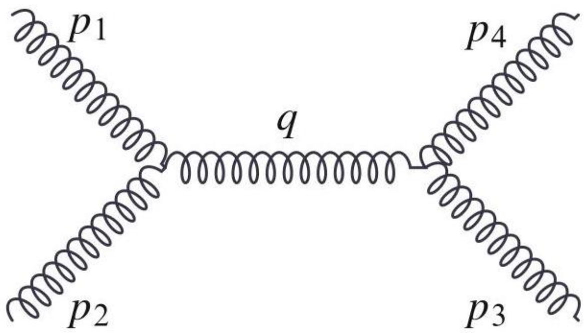


Fig. 3 *t* channel

图 3 *t* 道

Fig. 4 s channel

图 4 s 道

The four-graviton vertex corresponds to

四引力子顶点对应于

$$
\begin{aligned}
V^{\mu\nu,\rho\sigma,\alpha\beta,\eta\lambda}_{(p1,p2,p3,p4)} = i2s\kappa\mathcal{S}\Bigg\{&\frac{(2+n)(p_3\cdot p_4)}{n^2}\left[\frac{g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}g^{\eta\lambda}}{4n^2}-\frac{g^{\mu\rho}g^{\alpha\beta}g^{\eta\lambda}g^{\nu\sigma}}{4n}+g^{\mu\rho}g^{\alpha\beta}g^{\eta\sigma}g^{\nu\lambda}\right]- \\
&-\frac{1}{2}(p_3\cdot p_4)g^{\mu\eta}g^{\rho\lambda}g^{\alpha\upsilon}g^{\sigma\beta}+\frac{(2+n)(p_3\cdot p_4)g^{\mu\eta}g^{\rho\alpha}g^{\upsilon\lambda}g^{\sigma\beta}}{2n^2}-\frac{(2+n)(p_3\cdot p_4)g^{\mu\upsilon}g^{\rho\eta}g^{\alpha\beta}g^{\sigma\lambda}}{n^3}- \\
&-\frac{(p_3\cdot p_4)}{n}\left[\frac{g^{\mu\upsilon}g^{\rho\sigma}g^{\alpha\eta}g^{\beta\lambda}}{4n}+g^{\mu\upsilon}g^{\rho\eta}g^{\alpha\sigma}g^{\beta\lambda}-ng^{\mu\rho}g^{\alpha\upsilon}g^{\eta\sigma}g^{\beta\lambda}+\frac{g^{\mu\rho}g^{\alpha\eta}g^{\upsilon\sigma}g^{\beta\lambda}}{4}\right]+ \\
&+g^{\mu\eta}g^{\alpha\sigma}g^{\beta\lambda}p_3^{\upsilon}p_4^{\rho}+\frac{2+n}{2n^2}\left[g^{\mu\rho}g^{\alpha\beta}g^{\eta\lambda}p_3^{\sigma}p_4^{\upsilon}-\frac{g^{\mu\upsilon}g^{\alpha\beta}g^{\eta\lambda}p_3^{\rho}p_4^{\sigma}}{n}+2g^{\mu\alpha}g^{\eta\lambda}g^{\upsilon\beta}p_3^{\rho}p_4^{\sigma}\right]- \\
&-\frac{1}{2}g^{\mu\rho}g^{\alpha\eta}g^{\beta\lambda}p_3^{\sigma}p_4^{\upsilon}+\frac{g^{\mu\upsilon}g^{\alpha\eta}g^{\beta\lambda}p_3^{\rho}p_4^{\sigma}}{2n}-g^{\mu\alpha}g^{\eta\upsilon}g^{\beta\lambda}p_3^{\rho}p_4^{\sigma}-2\frac{g^{\mu\alpha}g^{\rho\beta}g^{\eta\lambda}p_3^{\upsilon}p_4^{\sigma}}{n}+ \\
&+2g^{\mu\rho}g^{\alpha\lambda}g^{\eta\sigma}p_3^{\upsilon}p_4^{\beta}-2\frac{g^{\mu\rho}g^{\alpha\sigma}g^{\eta\lambda}p_3^{\upsilon}p_4^{\beta}}{n}+2g^{\mu\eta}g^{\rho\lambda}g^{\alpha\upsilon}p_3^{\sigma}p_4^{\beta}-2\frac{g^{\mu\upsilon}g^{\rho\eta}g^{\alpha\lambda}p_3^{\sigma}p_4^{\beta}}{n}+ \\
&+2\frac{g^{\mu\nu}g^{\rho\alpha}g^{\eta\lambda}p_3^{\sigma}p_4^{\beta}}{n^2}-2\frac{g^{\mu\eta}g^{\rho\alpha}g^{\nu\lambda}p_3^{\sigma}p_4^{\beta}}{n}+\frac{g^{\mu\nu}g^{\rho\sigma}g^{\alpha\eta}p_3^{\lambda}p_4^{\beta}}{2n^2}-\frac{g^{\mu\nu}g^{\rho\eta}g^{\alpha\rho}p_3^{\lambda}p_4^{\beta}}{n} \\
&+g^{\mu\rho}g^{\alpha\upsilon}g^{\eta\sigma}p_3^{\lambda}p_4^{\beta}-\frac{g^{\mu\rho}g^{\alpha\eta}g^{\upsilon\sigma}p_3^{\lambda}p_4^{\beta}}{2n}-2\frac{g^{\mu\alpha}g^{\eta\sigma}g^{\upsilon\beta}p_3^{\rho}p_4^{\lambda}}{n}-\frac{g^{\mu\upsilon}g^{\rho\sigma}g^{\alpha\beta}p_3^{\eta}p_4^{\lambda}}{n^3}+ \\
&+\frac{g^{\mu\rho}g^{\alpha\beta}g^{\upsilon\sigma}p_3^{\eta}p_4^{\lambda}}{n^2}-2\frac{g^{\mu\rho}g^{\alpha\sigma}g^{\upsilon\beta}p_3^{\eta}p_4^{\lambda}}{n}+2\frac{g^{\mu\upsilon}g^{\rho\alpha}g^{\sigma\beta}p_3^{\eta}p_4^{\lambda}}{n^2}- \\
&-2\frac{g^{\mu\rho}g^{\alpha\beta}g^{\eta\sigma}p_3^{\upsilon}p_4^{\lambda}}{n}+2\frac{g^{\mu\upsilon}g^{\rho\eta}g^{\alpha\beta}p_3^{\sigma}p_4^{\lambda}}{n^2}\Bigg\}.
\end{aligned}
\tag{171}
$$

where $\mathcal{S}$ is a shorthand for a double symmetrization, namely:

其中 $\mathcal{S}$ 是双重对称化的简写，即:

1. A summation over all momentum-index combinations $(p_1,\mu\upsilon;\ p_2,\rho\sigma\ ;;\ p_3,\alpha\beta; p_4,\eta\lambda)$

1. 对所有动量-指标组合求和 $(p_1,\mu\upsilon;\ p_2,\rho\sigma\ ;;\ p_3,\alpha\beta; p_4,\eta\lambda)$

2. A symmetrization of each pair on indices $\mu\upsilon,\rho\sigma,\alpha\beta$, and $\eta\lambda$

2. 对每一对指标进行对称化 $\mu\upsilon,\rho\sigma,\alpha\beta$, and $\eta\lambda$

The fact that UG perturbatively expanded around Minkowski space is Lorentz invariant and the graviton polarizations are the same as in GR means, by the standard analysis - see [34] - that the on-shell three-point amplitudes vanish on-shell for real momenta. Since the little group scaling operates in UG precisely in the same manner as in GR, it is plain that for conserved complex momenta, the on-shell nonvanishing three-point amplitudes are the same in both theories but, perhaps, for a global constant. By explicit computation of the corresponding diagrams, it can be shown that the constant in question is the same in both theories, as it becomes the fact that the classical Newton constant is also the same in both theories; see section ”Physical Sources.”

在闵氏空间周围微扰展开的 UG 具有洛伦兹不变性，且其引力子极化与广义相对论 GR 中的一致，根据标准分析 (参见文献 [34])，这意味着实动量下的在壳三点振幅在壳为零。由于 UG 中的小群标度变换与 GR 中的作用方式完全相同，很明显对于守恒复动量，两种理论中的非零在壳三点振幅是一致的，至多相差一个整体常数。通过对对应图的显式计算可以证明，两种理论中的该常数是相同的，这是因为两种理论中的经典牛顿引力常数也相同；参见“物理源”一节。

Having clarified the three-graviton diagrams, one can further consider four-graviton tree amplitudes. Since the pure four-vertex diagram vanishes, only three types of diagrams involve four external gravitons. These are the well-known s, t , and u channels; see Figs. 2,3, and 4.

在厘清三引力子图后，我们可以进一步考虑四引力子树振幅。由于纯四顶点图为零，仅存在三类包含四个外引力子的图。即大家熟知的 s, t 和 u 道；参见图 2、图 3 和图 4。

Following [13], the polarization tensors for the gravitons are written in terms of those of the gluon:

根据文献 [13]，引力子的极化张量可以用胶子的极化张量表示为:

$$\varepsilon^{-}_{\mu\nu} = \varepsilon^{-}_{\mu}\varepsilon^{-}_{\nu} \to \varepsilon^{-}_{a\dot{a},b\dot{b}} = \varepsilon^{-}_{a\dot{a}}\varepsilon^{-}_{b\dot{b}} \text{ and } \varepsilon^{+}_{a\dot{a},b\dot{b}} = \varepsilon^{+}_{a\dot{a}}\varepsilon^{+}_{b\dot{b}}. \tag{172}$$

The resulting amplitudes are then

得到的振幅为

$$\begin{aligned}\mathcal{A}_s\left(1^-2^-;3^+4^+\right) &= \varepsilon_1^{-\mu_1}\varepsilon_1^{-\upsilon_1}\varepsilon_2^{-\mu_2}\varepsilon_2^{-\upsilon_2}V^{\mu_1\upsilon_1,\mu_2\upsilon_2,\alpha\beta}_{(p1,p2,q)}P_{\alpha,\beta,\rho,\sigma}V^{\rho\sigma,\mu_3\upsilon_3,\mu_4\upsilon_4}_{(p,p3,p4)}\varepsilon_3^{-\mu_3}\varepsilon_3^{-\upsilon_3}\varepsilon_4^{-\mu_4}\varepsilon_4^{-\upsilon_4}\\ &= -\frac{i4\kappa^2(e_1\cdot p_2)^2(e_2\cdot e_3)^2(e_4\cdot p_2)^2}{s^2},\end{aligned} \tag{173}$$

$$\begin{aligned}\mathcal{A}_t\left(1^-3^+;2^-4^+\right) &= \varepsilon_1^{-\mu_1}\varepsilon_1^{-\upsilon_1}\varepsilon_3^{-\mu_3}\varepsilon_3^{-\upsilon_3}V^{\mu_1\upsilon_1\upsilon_1,\mu_3\upsilon_3,\alpha\beta}_{(p1,p3,q)}P_{\alpha,\beta,\rho,\sigma}V^{\rho\sigma,\mu_2\upsilon_2,\mu_4\upsilon_4}_{(p,p2,p4)}\varepsilon_2^{-\mu_2}\varepsilon_2^{-\upsilon_2}\varepsilon_4^{-\mu_4}\varepsilon_4^{-\upsilon_4}\\ &= 0\,,\end{aligned}$$

(174)

$$\begin{aligned}\mathcal{A}_u\left(1^-4^+;2^-3^+\right) &= \varepsilon_1^{-\mu_1}\varepsilon_1^{-\upsilon_1}\varepsilon_4^{-\mu_4}\varepsilon_4^{-\upsilon_4}V^{\mu_1\upsilon_1,\mu_4\upsilon_4,\alpha\beta}_{(p1,p4,q)}P_{\alpha,\beta,\rho,\sigma}V^{\rho\sigma,\mu_2\upsilon_2,\mu_3\upsilon_3}_{(p,p2,p3)}\varepsilon_2^{-\mu_2}\varepsilon_2^{-\upsilon_2}\varepsilon_3^{-\mu_3}\varepsilon_3^{-\upsilon_3}\\ &= -\frac{i4\kappa^2(e_1\cdot p_2)^2(e_2\cdot e_3)^2(e_4\cdot p_2)^2}{u^2}.\end{aligned}$$

(175)

In Eqs. (173),(174), and (175), as usual, $s = p_1 + p_2$ and $u = p1 + p3$. These amplitudes are diagram to diagram the same as those for GR.

在式 (173)、(174) 和 (175) 中，和通常一样，$s = p_1 + p_2$ and $u = p1 + p3$ 。这些振幅逐图与 GR 中的振幅完全一致。

This equivalence is maintained for the tree diagrams with five external gravitons, as shown in [29].

如文献 [29] 所示，五个外引力子的树图也保持了这种等价性。

To summarize, it has been shown that the maximal helicity violating (MHV), three, four, and five graviton tree amplitudes give the same contribution in GR and UG. Moreover, this result holds for each diagram independently and not only for the total amplitude. Therefore, at least at tree level, and with three, four, or five external legs, the MHV contribution to the S matrix for pure gravity without coupling to other fields is the same in both theories.

总而言之，已有证明表明最大螺旋度破坏 (MHV) 的三、四、五引力子树振幅在 GR 和 UG 中给出相同的贡献。此外，该结果对每个图独立成立，而非仅对总振幅成立。因此，至少在树图水平，且对于三、四、五个外腿的情况，不耦合其他场的纯引力 S 矩阵的 MHV 贡献在两种理论中是相同的。

A remarkable fact is that all the terms that involve the double and triple poles in the propagator of UG (167) do not contribute to any diagram.

一个值得注意的事实是，UG 传播子 (167) 中所有包含双极点和三极点的项对任何图都没有贡献。

This cancelation is not trivial, and it came as a surprise in the original work. Indeed, in UG, one obtains the following non-zero result:

这种抵消并非平凡，在最初的研究中这是一个出人意料的结果。实际上，在 UG 中可以得到如下非零结果:

$$k_\alpha k_\beta V^{\mu\nu,\rho\sigma,\alpha\beta}_{(p,q,k)} \varepsilon_{1\mu\nu}(p)\varepsilon_{2\rho\sigma}(q) = i\kappa(p\cdot q)(p\cdot\varepsilon_2)(q\cdot\varepsilon_1)(\varepsilon_1\cdot\varepsilon_2), \tag{176}$$

when $k = -p - q$ is off-shell and the polarizations with well-defined helicity $\varepsilon_{1\mu\upsilon}(p) = \varepsilon_{1\mu}(p)\varepsilon_{1\upsilon}(p)$ and $\varepsilon_{2\rho\sigma}(q) = \varepsilon_{2\rho}(q)\varepsilon_{2\sigma}(q)$ are arbitrary. On the other hand, the computation of the corresponding object in GR yields a vanishing result as a consequence of Diff invariance. It could be that the gauge invariance of UG and the fact that the only physical degrees of freedom in perturbative UG are gravitons are the ones to be held responsible, at least partially, for this cancellation.

当 $k = -p - q$ 离壳，且具有确定螺旋度 $\varepsilon_{1\mu\upsilon}(p) = \varepsilon_{1\mu}(p)\varepsilon_{1\upsilon}(p)$ 和 $\varepsilon_{2\rho\sigma}(q) = \varepsilon_{2\rho}(q)\varepsilon_{2\sigma}(q)$ 的极化任意时。另一方面，由于微分同胚不变性，GR 中对应量的计算结果为零。UG 的规范不变性，以及微扰 UG 中仅有的物理自由度是引力子这一事实，至少部分是这种抵消的原因。

One-Loop Unimodular Gravity

单圈幺模引力

Until this point, the comparison of UG and GR has considered the EM and their solutions and tree diagrams. Both UG and GR propagate the same number of degrees of freedom. Furthermore, the tree diagrams explored in section "Tree Diagrams" coincide with those of GR despite the notable differences for the propagators in both theories. The reader may then be inclined to think that it must be the case that any tree-level computation might be equivalent for both theories. Computations in section "Tree Diagrams" explicitly show that this is far from trivial.

至此，我们对幺模引力 (UG) 和广义相对论 (GR) 的比较已经涵盖了场方程、它们的解以及树图。幺模引力和广义相对论传播的自由度数量相同。此外，尽管两种理论的传播子存在显著差异，“树图”一节中研究的树图仍与广义相对论的树图一致。因此读者可能会倾向于认为，两种理论的任何树级计算都必然等价。但“树图”一节的计算明确表明，事实远非如此。

In any case, any potential equivalence at the tree level need not extrapolate to loop-level computations. In the path integral approach to the quantization of UG, the path integral measure must incorporate the fact that the gauge group is not Diff but WTDiff, as discussed, for example, in [35].

无论如何，树级的潜在等价性并不需要推广到圈级计算。正如例如文献 [35] 中所讨论的，在幺模引力量子化的路径积分方法中，路径积分测度必须考虑规范群不是微分同胚群 (Diff) 而是加权保体积微分同胚群 (WTDiff) 这一事实。

Therefore, while on-shell states match for both UG and GR, since the two theories do not share gauge groups, while GR gravitons running on loops being off-shell need not be traceless, UG gravitons are traceless even inside loops. Simply put, loops run over different states in each case. Therefore, in perturbation theory, at least a priori, one lacks a reason to expect the cancellations of these differences at higher orders. From the reasons above, one could expect potential differences to arise between UG and GR at the one-loop level.

因此，尽管幺模引力和广义相对论的在壳态一致，但两种理论的规范群并不相同: 广义相对论中在圈上运动的离壳引力子不必是无迹的，而幺模引力的引力子即使在圈内部也是无迹的。简单来说，两种情况中圈遍历的是不同的态。因此在微扰论中，至少先验来看，我们没有理由预期这些差异会在高阶抵消。基于上述原因，我们可以预期幺模引力和广义相对论在单圈水平会出现潜在差异。

In the last part of this chapter, a particular formalism [15] that allows dealing with the involved gauge sector of UG in one-loop computations is presented.

本章最后一部分将介绍一种特殊形式体系 [15]，可用于处理幺模引力单圈计算中复杂的规范区。

Consider the background field expansion. Once again, the metric is split field into the sum of a classical and a quantum part:

考虑背景场展开。我们再次将度规拆分为经典部分和量子部分之和:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}. \tag{177}$$

Additionally, assume the measure of the path integral is shift-invariant, i.e.,

此外，假设路径积分测度具有平移不变性，即:

$$\int [\mathcal{D}g_{\mu\nu}] \exp [iS[g] + T \cdot g] = \int [\mathcal{D}h_{\mu\nu}] \exp [iS[h] + T \cdot g], \tag{178}$$

where $T \cdot g$ represents the sources that are added to the partition function to obtain field expectation values by functional differentiation

其中 $T \cdot g$ 表示为通过泛函微分得到场期望值而添加到配分函数中的源

$$T \cdot g = \int d^4x \sqrt{g} T^{\mu\nu} g_{\mu\nu}. \tag{179}$$

The beauty of this method is that it allows fixing the gauge of the quantum part while preserving the gauge invariance of the background. Then all computations are invariant under gauge transformations of the background. [16] The same happens for the counterterms.

该方法的优点在于，它可以在固定量子部分规范的同时保留背景的规范不变性。因此所有计算都在背景的规范变换下保持不变。[16] counterterms(抵消项) 也是如此。

As it happens for other gauge theories, gauge freedom introduces an indeterminacy in the path integral. To resolve this, one often uses the DeWitt-Feynman-Faddeev-Popov method [37, 38]. This adds to the original action, a part coming from the gauge fixing and certain ghost content representing the Jacobian of the gauge fixing condition.

和其他规范理论一样，规范自由度会给路径积分带来不确定性。为了解决这个问题，人们通常使用 DeWitt-Feynman-Faddeev-Popov 方法 [37, 38]。该方法在原作用量中添加了来自规范固定的部分，以及代表规范固定条件雅可比的鬼场项。

$$S \to S + S_{\mathrm{G,F}} + S_{\mathrm{Gh}} \tag{180}$$

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[15] This choice is by no means unique, and there is an ongoing discussion on whether different approaches to the quantization might yield different results [36]. Despite its interest, this possible inequivalence will not be further discussed here.

[15] 这种选择绝非唯一，目前关于不同量子化方法是否会得出不同结果仍存在讨论 [36]。尽管这种潜在不等价性很有意思，本文不会对此展开进一步讨论。

[16] It is in this sense that this approach is covariant.

[16] 从这个意义上来说，该方法是协变的。

Finding a gauge fixing condition for the quantum WTDiff that preserves the background's WTDiff invariance is an open problem [39]. Invariance under WTDiff implies, in particular, that the generators of the TDiff sector have to be transverse.

找到满足背景 WTDiff 不变性的量子 WTDiff 规范固定条件是一个开放问题 [39]。WTDiff 下的不变性尤其意味着，TDiff(保体积微分同胚) 区的生成元必须是横截的。

This is the main reason for the more involved nature of the gauge sector of UG. Nevertheless, these complications can be tackled at the expense of introducing new [17] bosonic ghost fields.

这就是幺模引力规范区更为复杂的主要原因。不过，这些复杂问题可以通过引入新的 [17] 玻色鬼场来解决。

To resolve the technicalities discussed above, one can consider the BRST approach as an alternative to the usual gauge fixing procedure. The basic ideas for the unfamiliar reader are summarized below. It was found that the gauge fixed action Eq. (180) is invariant under a BRST transformation [40,41]. This implies that the physical content of the gauge theory is given by the cohomology class of the BRST nilpotent operator $\mathfrak{s}$. In the case at hand, one can split

为了解决上述技术问题，我们可以将 BRST 方法作为常规规范固定程序的替代方案。下文为不熟悉该方法的读者总结了基本思路。研究发现，式 (180) 的规范固定作用量在 BRST 变换下保持不变 [40,41]。这意味着规范理论的物理内容由 BRST 幂零算子 $\mathfrak{s}$ 的上同调类给出。在我们讨论的情况中，可以拆分

$$\mathfrak{s} = \mathfrak{s}_W + \mathfrak{s}_D, \quad \text{where} \tag{181}$$

$$\mathfrak{s}\bar{g}_{\mu\nu} = 0, \tag{182}$$

$$\mathfrak{s}h_{\mu\nu} = \mathfrak{s}_D h_{\mu\nu} + \mathfrak{s}_W h_{\mu\nu} = \mathcal{L}_{c^\mu}\left(\bar{g}_{\mu\nu} + h_{\mu\nu}\right) + 2c\left(\bar{g}_{\mu\nu} + h_{\mu\nu}\right). \tag{183}$$

The action is given in terms of the original Lagrangian density by

作用量由原拉格朗日密度给出，形式为:

$$S_{\text{BRST}} = \int d^4x\mathcal{L} + \mathfrak{s}\Psi. \tag{184}$$

Now, because the generator of the volume-preserving diffeomorphism is transverse, so must be the associated ghost field c^μ :

现在，由于保体积微分同胚的生成元是横向的，关联的鬼场 c^μ 也必须是横向的:

$$D_\mu c^\mu = 0, \tag{185}$$

where the transversality condition is given by the Weyl covariant derivative.

其中横向性条件由外尔协变导数给出。

The easiest way to implement this is using a projector [21,39] Π^{μ}_{υ} acting on an unrestricted c^{υ} . This introduces a $U(1)$ invariance:

实现这一点最简单的方法是利用投影算符 [21,39] Π^{μ}_{υ} 作用在不受限的 c^{υ} 上。这会引入一个 $U(1)$ 不变性:

$$c^{\upsilon} \to D^{\upsilon} f \tag{186}$$

A general treatment of such an algebra requires introducing the BV quantization techniques [42,43]. Nevertheless, in this case, the procedure can be carried out, to the one-loop level, by enlarging the ghost content of the theory [44-46]. Let us now give the guidelines to do so.

对这类代数的一般处理需要引入 BV 量子化技术 [42,43]。但在本情况中，该过程可以通过扩充理论的鬼场内容 [44-46]，推广至单圈水平。下面我们给出具体操作的大致框架。

The first step is to include the necessary set of anti-ghost and auxiliary fields that close the algebra.

第一步是引入封闭代数所需的全套反鬼场与辅助场。

$$\begin{gathered} h_{\mu\upsilon}{}^{(0,0)}, c_{\mu}^{(1,1)}, b_{\mu}^{(1,-1)}, f_{\mu}^{(0,0)}, \phi^{(0,2)} \\ \pi^{(1,-1)}, \pi'^{(1,1)}, \bar{c}^{(0,-2)}, c'^{(0,0)} \\ c^{(1,1)}, b^{(1,-1)}, f^{(0,0)}, \end{gathered} \tag{187}$$

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[17] Not present for GR.

[17] 广义相对论中不存在该项。

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where, in the label (n, m) , n denotes the Grassmann number (modulo 2) and m is the ghost number. The first line corresponds to the physical graviton and the ghost fields that one would naively need for Diff. In addition, ϕ corresponds to the $U(1)$ transformation. The second line has the ghost content to fix that $U(1)$, and the last line corresponds to the Weyl invariance.

其中标记里 (n, m) , n 表示格拉斯曼数 (模 2)，m 是鬼数。第一行对应物理引力子，以及微分同胚变换直观上需要的鬼场。另外，ϕ 对应 $U(1)$ 变换。第二行是固定 $U(1)$ 所需的鬼场，最后一行对应外尔不变性。

This is enough to make $\mathfrak{s}$ nil-potent to the one-loop level, ensuring BRST invariance of the total action. For completeness, the counterterm reads

这足以保证在单圈水平 $\mathfrak{s}$ 是幂零的，从而确保总作用量具有 BRST 不变性。为完整起见，counterterm(抵消项) 写为

$$S_{\text{BRST}}^{\text{TDiff}} + S_{\text{BRST}}^{\text{Weyl}} = \int d^n x b^\mu \left(\Box^2 c_\mu^{(1,1)} - 2R_{\mu\rho}\nabla^\rho\nabla^\nu c_\nu^{(1,1)} - \Box R_\mu{}^\rho c_\rho^{(1,1)} - \right.$$

$$-2\nabla_\sigma R_\mu{}^\rho \nabla^\sigma c_\rho^{(1,1)} - \left. R_{\mu\rho}R^{\rho\nu}c_\nu^{(1,1)}\right) - \bar{c}^{(0,-2)}\Box\phi^{(0,2)} + \pi^{(1,-1)}\Box\pi'^{(1,1)} -$$

$$-\frac{1}{\rho_1}\left(F_\mu F^\mu + \nabla_\mu c'^{(0,0)}\nabla^\mu c'^{(0,0)} + 2F_\mu\nabla^\mu c'^{(0,0)}\right) - f^{(0,0)}\Box f^{(0,0)} + \frac{\alpha}{2}f^{(0,0)}\Box h +$$

$$+\frac{\alpha}{2}h\Box f^{(0,0)} + 2n\alpha b^{(1,-1)}\Box c^{(1,1)}, \text{ where } F_\mu \equiv \nabla^\upsilon h_{\mu\upsilon} - \frac{1}{n}\nabla_\mu h\,. \tag{188}$$

Once the ghost content issue is solved, one can apply the Schwinger-DeWitt proper time expansion [18] to calculate the divergences of UG [21]. Let us present here the main ideas. The sum of the divergent parts [19] corresponds to [21]

解决鬼场内容的问题后，就可以应用施温格-德维特固有时间展开 [18] 计算单模引力 (UG) 的发散项 [21]。我们在这里介绍核心思路。发散部分的和 [19] 对应 [21]

$$W_\infty = \frac{1}{16\pi^2}\frac{1}{n-4}\int d^n x\left(\frac{119}{90}R_{\mu\upsilon\alpha\beta}R^{\mu\upsilon\alpha\beta} + \left(\frac{1}{6\alpha^2} - \frac{359}{90}\right)R_{\mu\upsilon}R^{\mu\upsilon} + \right.$$

$$\left. + \frac{1}{72}\left(22 - \frac{3}{\alpha^2}\right)R^2\right) \tag{189}$$

Start by focusing on the issue of on-shell renormalizability. It is known that although GR is one-loop finite in the absence of a CC, this property is lost in its presence. The on-shell counterterm, in this case, was obtained in [47], and it amounts to a renormalization of the CC and is proportional to

我们先来关注在壳可重整化性问题。已知，尽管广义相对论在不存在宇宙学常数 (CC) 时单圈有限，但当存在宇宙学常数时，这一性质就会消失。这种情况下，在壳抵消项已在文献 [47] 中得到，它对应宇宙学常数的重整化，且正比于

$$W_\infty^{GR} \equiv \frac{1}{16\pi^2(n-4)}\int \sqrt{g}d^4x\left(\frac{53}{45}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - \frac{1142}{135}\Lambda^2\right). \tag{190}$$

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[18] See [49] for an introduction to these techniques.

[18] 关于这些技术的介绍可参见 [49]。

[19] The result in Eq. (189), which uses the gauge choice of [21], could seem to imply that one cannot make the terms proportional to R and R^2 vanish, but as (190) shows, one can remove these terms employing a two-parameter gauge fixing [48].

[19] 式 (189) 的结果采用了文献 [21] 的规范选择，看似意味着无法消去正比于 R 和 R^2 的项，但正如式 (190) 所示，我们可以通过双参数规范固定 [48] 消去这些项。

Since the main attractive feature of UG is precisely the different rôle that the CC plays in contrast to GR, it is interesting to see what happens here with the renormalization group flow when the counterterm is taken to be on-shell so that all external legs correspond to physical states. In that case, the EM for the $g = 1$ fixed background are the traceless Einstein equations

由于单模引力最吸引人的特点正是宇宙学常数所扮演的角色与广义相对论不同, 因此当抵消项取在壳形式 (所有外腿都对应物理态) 时，考察重整化群流的行为十分有意义。在这种情况下，$g = 1$ 固定背景的爱因斯坦运动方程是无迹爱因斯坦方程

$$R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = 0, \tag{191}$$

which, altogether with Bianchi identities, imply the following for the operators appearing in the effective action:

结合比安基恒等式，可得有效作用量中出现的算符满足如下关系:

$$R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = E_4, \tag{192}$$

$$R_{\mu\nu} R^{\mu\nu} = \frac{1}{4} R^2 \tag{193}$$

$$R = \text{ constant.} \tag{194}$$

The first line is nothing more than the Gauss-Bonnet theorem when considering the EM. E_4 is thus the Pfaffian (the Euler density), whose integral gives the Euler characteristic of the manifold.

考虑爱因斯坦运动方程时，第一行正是高斯-博内定理。因此 E_4 是普法夫式 (欧拉密度)，它的积分给出流形的欧拉示性数。

By using these, the on-shell effective action can be cast in the form

利用上述关系，在壳有效作用量可以写为如下形式

$$W_\infty^{\text{on-shell}} = \frac{1}{16\pi^2} \frac{1}{n-4} \int d^n x \left(\frac{119}{90} E_4 - \frac{83}{120} R^2 \right). \tag{195}$$

The contribution of the CC to the effective action is a non-dynamical quantity. Furthermore, this contribution does not couple to the metric because the $\sqrt{g}$ factor in the integration measure is absent. This implies one can disregard this term since it will not contribute to any correlator involving physical fields. One can therefore conclude that in this case, there is no renormalization of the CC, and its peculiar status in UG is preserved through quantum corrections.

宇宙学常数对有效作用量的贡献是一个非动力学量。此外，由于积分测度中不存在 $\sqrt{g}$ 因子，该贡献不与度规耦合。这意味着我们可以忽略这一项，因为它不会对任何涉及物理场的关联函数产生贡献。因此可以得出结论: 在该情况下，宇宙学常数不存在重整化，它在幺模引力中的特殊地位在量子修正下得以保留。

Indeed, this effect is not specific to one-loop computations. The bare value of the CC is protected, and quantum corrections do not modify it.

事实上，这一效应并不局限于单圈计算。宇宙学常数的裸值受到保护，量子修正不会改变它。

It could be thought that this effect is just a gauge artifact of our background choice $\gamma = 1$. However, it can be easily argued that this is not the case. As previously commented in this work, to obtain the effective action for an arbitrary background from the one with a unimodular background metric, it is enough to make a change of variables so that

有人可能会认为这个效应只是我们背景选择 $\gamma = 1$ 带来的规范赝效应。但我们可以很容易地说明事实并非如此。正如本文之前所述，若要从幺模背景度量的结果得到任意背景的有效作用量，只需进行一次变量变换，使得:

$$\gamma_{\mu\upsilon} = g^{-\frac{1}{n}} g_{\mu\upsilon}. \tag{196}$$

This transformation is available as long as a conformal anomaly is not generated. This has been argued to be the case since a regularization scheme exists in which the anomaly vanishes [50]. Although this argument bears similarity to the classic work of [51], here it is applied to a tautological symmetry (what Duff [52] calls pseudo-Weyl invariance) because there is a field redefinition in which the symmetry disappears. In this case, the consensus is [52] that there is no anomaly. Nevertheless, the final word on this point is still to be said.

只要不产生共形反常，这个变换就是成立的。已有研究证明这一情况是成立的，因为存在一个正则化方案可以让反常消失 [50]。尽管这一论证与文献 [51] 的经典工作相似，但本文中将它应用于一种重言对称性 (即达夫 [52] 所说的伪外尔不变性)，因为存在可以让该对称性消失的场重定义。在这个问题上，学界的共识是 [52] 不存在反常。尽管如此，关于这一点的盖棺定论仍有待后续研究给出。

When doing this, one can see that the real reason for the CC not being renormalized is indeed the presence of Weyl invariance in the formalism, which protects the appearance of any mass scale in the effective action and, as a consequence, in the expectation value of the EM. Therefore, the argument holds, and the CC is protected and fixed to its bare value all along the renormalization group flow and at any loop order.

完成这一变换后可以发现，宇宙学常数不被重整化的真正原因确实是形式体系中存在外尔不变性，它保护了有效作用量中不会出现任何质量标度，相应地，能量动量张量的期望值中也不会出现质量标度。因此，上述论证成立，在整个重整化群流中、任意圈阶下，宇宙学常数都受到保护，始终固定为它的裸值。

Summary

摘要

This work is devoted to an introduction to unimodular gravity (UG).

本文专门介绍幺模引力 (UG)。

The primary motivation to consider UG is the rôle of the cosmological constant in this theory. The main difference in this respect is that a constant vacuum energy does not weigh at all and does not induce a cosmological constant.

研究幺模引力的核心动机源于宇宙学常数在该理论中的作用。这一方面的主要区别在于: 常数真空能量在该理论中完全不产生引力，也不会诱导出宇宙学常数。

The most direct path to UG is to start by demanding the unitarity of the linear, spin-two field theory. There are only two solutions: one is the Fierz-Pauli theory and the other is UG.

得到幺模引力最直接的思路是从要求线性自旋 2 场论的幺正性出发，仅存在两种解: 一种是菲尔斯-保利理论，另一种就是幺模引力。

After this, a general discussion on the fully non-linear EM and their solutions has been presented. The coupling to matter sources is discussed in detail in section ”Physical Sources.” The static potential agrees precisely with the one of general relativity (GR). It can also be argued at the non-linear level that there is an equivalence of sorts between UG and GR with some cosmological constant, which is determined by the boundary conditions of the equations of motion and is not related to the presence of any vacuum energy.

在此之后，本文对完全非线性的场方程及其解进行了全面讨论，在「物理源」一节详细讨论了与物质源的耦合。其静态势与广义相对论 (GR) 的结果完全一致。在非线性层面也可以证明，幺模引力与带有某宇宙学常数的广义相对论存在某种等价性，该宇宙学常数由运动方程的边界条件确定，与任何真空能量的存在无关。

It has even been claimed that the equivalence from the classical perspective can be generalized to higher-derivative theories of gravity; see [36]. However, Birkhoff’s theorem is not valid in UG for a variety of reasons; see section ”No Birkhoff’s Theorem in UG.” This leads, in particular, to exponentially expanding vacuum solutions in cosmology discussed in section ”Unimodular Cosmology.” The last part of this chapter deals with a perturbative formulation of the path integral. Some improvement over GR in the ultraviolet is expected here because there is no integration over the conformal factor in UG. However, this fact is obscured in the Weyl invariant formulation presented in this work. Moreover, BRST symmetry is somewhat tricky due to the interference between Weyl and TDiff symmetries.

甚至有观点认为，经典视角下的等价性可以推广到高阶导数引力理论；参见文献 [36]。但由于多种原因，伯克霍夫定理在幺模引力中不成立，参见「幺模引力中不存在伯克霍夫定理」一节。这尤其引出了「幺模宇宙学」一节讨论的宇宙学中指数膨胀的真空解。本章最后一部分讨论了路径积分的微扰表述。由于幺模引力中不对共形因子积分，预期其紫外性质相比广义相对论有所改进，但这一特性在本文给出的外尔不变表述中并不明显。此外，由于外尔对称性与微分同胚对称性之间的干涉，BRST 对称性的处理较为复杂。

In doing so, tree-level amplitudes are discussed in section ”Tree Diagrams” (finding complete agreement with GR), followed by a brief discussion on the one-loop formulation of unimodular gravity.

在此框架下，「树图」一节讨论了树图振幅 (结果与广义相对论完全一致)，随后简要讨论了幺模引力的单圈表述。

The main result, first found in [13] and explicitly shown here, is that the cosmological constant is stable under radiative corrections.

本文明确展示了最初由文献 [13] 得到的核心结果: 宇宙学常数在辐射修正下是稳定的。

Some important topics that have not found a place in this work are the construction of a Hamiltonian for unimodular gravity, which can be used, for example, to study the Noether charges of the theory, an explicit discussion of the conformal factor in the path integral, and the York-Gibbons-Hawking boundary correction.

本文未能涵盖的重要主题包括: 幺模引力哈密顿量的构造 (可用于研究该理论的诺特荷)、路径积分中共形因子的明确讨论，以及约克-吉本斯-霍金边界修正。

Cross-References

交叉引用

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The Background Information About Perturbative Quantum Gravity

微扰量子引力背景知识介绍

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